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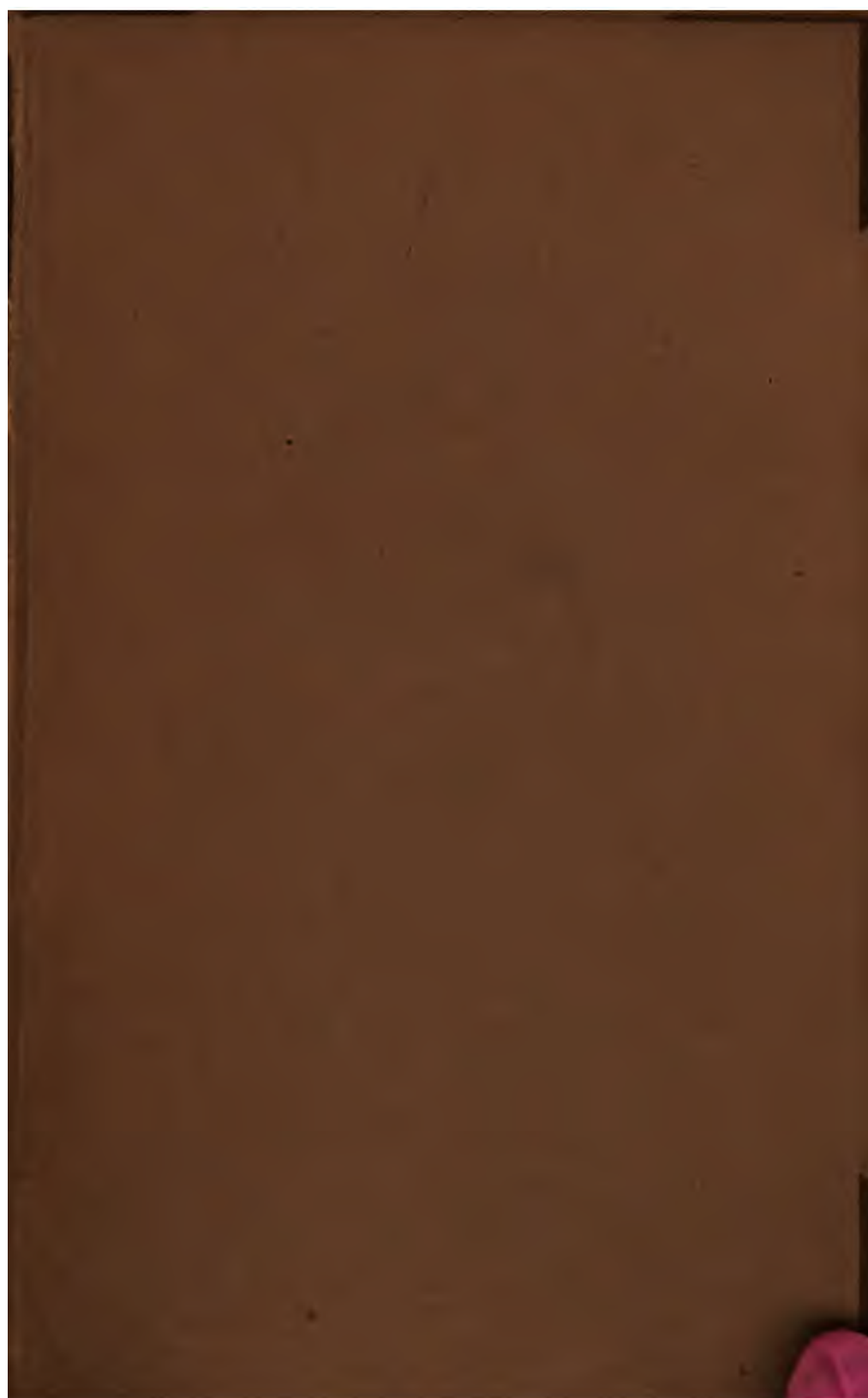
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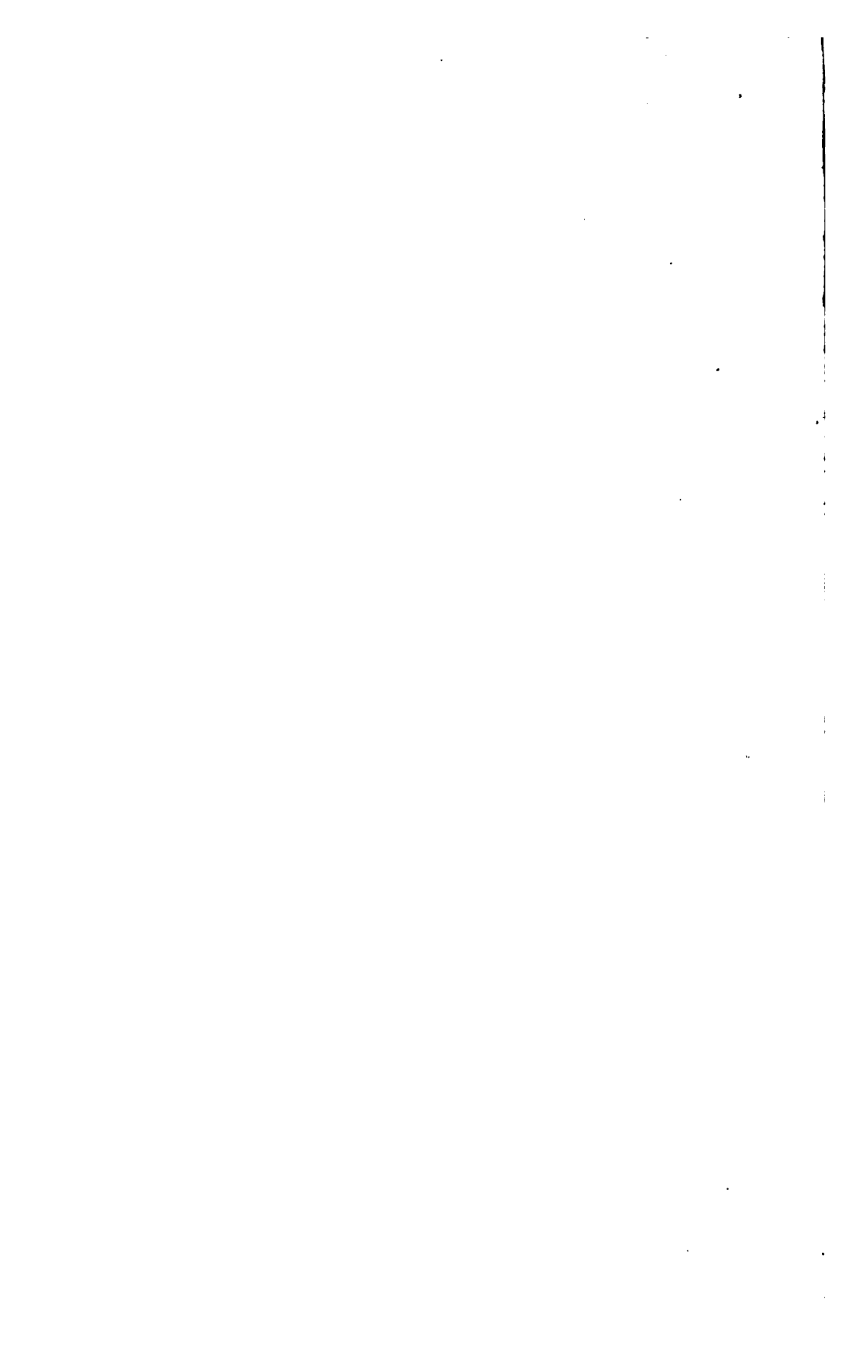
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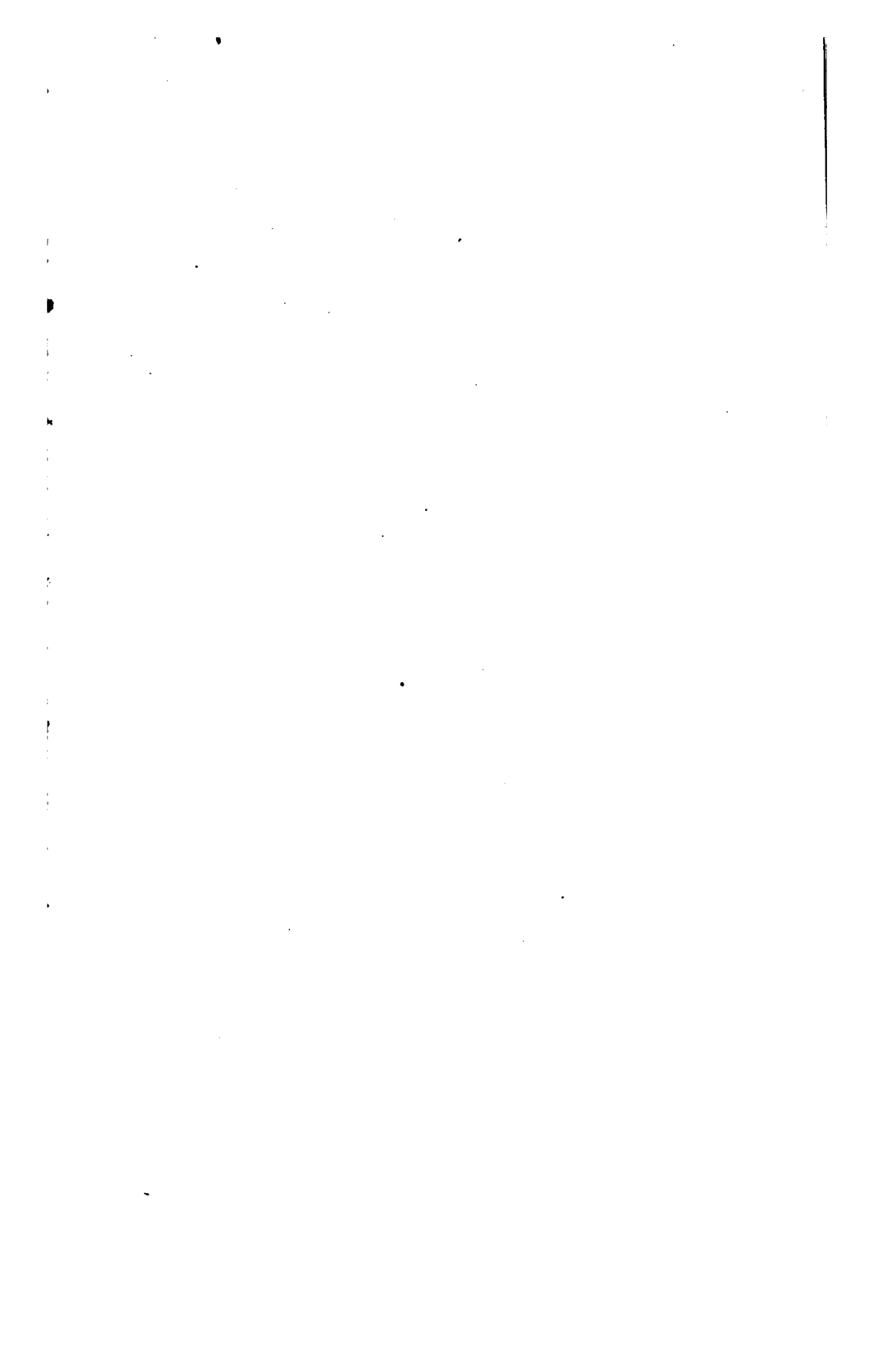
The Estate of

George Eastwood,

4 Feb., 1887.







A

T R E A T I S E

ON

DIFFERENTIAL EQUATIONS,

WITH A

COLLECTION OF EXAMPLES

ARRANGED IN CLASSES,

CORRESPONDING TO THE SEVERAL DIVISIONS OF THE
SUBJECT.

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From

the Estate of

George Eastwood,

4 Feb., 1887:

P R E F A C E.

THE object of the present Treatise is to collect and arrange such propositions relative to Differential Equations, as are most important in the investigations of Physical or Geometrical Science; and it is hoped that it will be found to contain all such parts of the subject as usually enter into the course of study pursued at this University.

The demonstrations have been obtained from various sources, and no pains have been spared to render them as simple, and, at the same time, as satisfactory as possible.

It has been attempted throughout, rather to extend *general* methods of solution, so as to make them available with a great number of equations, than to exhibit a multitude of particular artifices applicable only to particular cases.

In few subjects, perhaps, are examples more necessary to the student than in Differential Equations, and, accordingly, a considerable number have been introduced; with a view, however, to preserve the continuity of the work, only one example of each sort has been inserted in the text, the remainder being arranged in an Appendix, in classes corresponding to the natural divisions of the subject.

With respect to the notation, both differentials and differential coefficients have been used; and, previous to integration, an equation has always been expressed by means of the former. The reasons which induced the Author to retain the old notation $\frac{du}{dx}$, $\int u dx$, &c. were, first, because he believes it to be much more generally used than the other, and, secondly, because he was unwilling to adopt *a system* with equations between two variables, which he must have relinquished when treating total differential equations between three variables.

The notation $d_x u$, $\int_x u$, &c., appears to deny the existence or meaning of a differential, and under these circumstances, how is the equation $Pdx + Qdy + Rdz = 0$ to be expressed? It is true it has been written $P + Qd_x y + Rd_x z = 0$, but surely this is not reconcilable with either system.

The Author cannot but feel that the task he has undertaken is one of no small difficulty; and while he indulges a hope that his labors may not be considered altogether futile, he looks forward with deference to that frank criticism, which so many in this place are well qualified to afford him.

Cambridge,

Oct. 26th, 1832.

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ERRATA.

Page 18, line 12, for "when the arbitrary constant is supposed to vanish,"
read, *when a proper value is assigned to the arbitrary constant.*

Page 81, line 31, for $\frac{2me^{ms}}{m^2 - k^2}$, read $\frac{2ke^{ms}}{m^2 - k^2}$.

Page 94, line 10, read

$$x = \frac{e^{-\rho t}}{2} \cdot \left\{ \left(C'' - C' \frac{a}{\beta} \right) \cos \theta t + \left(C' + C'' \frac{a}{\beta} \right) \sin \theta t \right\}.$$

In some copies, page 127, line 24, for $B + pd$, read $B + pD$; and line 26, for
+ read —.

Page 139, in the integral of Ex. 1, for $bx + c$, read $\frac{y + c}{b}$.

Page 151, line 11, for C , read c .

PART I.

ON DIFFERENTIAL EQUATIONS

BETWEEN TWO VARIABLES.

CHAPTER I.

GENERAL PROPERTIES.

WE learn by the Differential Calculus, that constants may be eliminated from an equation between x and y , by differentiation; the result being an equation, which in its most general form contains x , y , and the differential coefficients of y taken with respect to x as the independent variable.

Suppose we have an equation

$$\phi(x, y, a, b, c \dots n) = 0 \dots (a),$$

containing n constants; then, representing the successive differential coefficients, taken with respect to x by accents, we might, by n differentiations, obtain the additional n equations

$$\phi'(x, y, a, b \dots n),$$

$$\phi''(x, y, a, b \dots n),$$

$$\dots \dots \dots$$

$$\phi^{(n)}(x, y, a, b \dots n);$$

which, with the original equation (a), which is called the

primitive, would give in all $(n + 1)$ equations, and therefore we know, by the rules of algebra, that the n quantities $a, b, c, \dots n$ might be eliminated between them, and the final result would be an equation of the form

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2} \dots \frac{d^ny}{dx^n}\right) = 0,$$

which is called a differential equation of the n^{th} order, because it contains the n^{th} differential coefficient of y , with respect to x .

The investigations connected with differential equations, between two variables, have for their ultimate object the finding of the primitive $\phi(x, y, a, b \dots n)$ from the differential equation

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2} \dots \frac{d^ny}{dx^n}\right) = 0.$$

It is obvious, that to do this n constants should be introduced, somehow connected with variables, because the primitive contained, or at any rate might contain, n constants; but as the particular values of these are not indicated by the differential equation, they are usually styled *arbitrary constants*, although their values may, for the most part, be determined from the nature of the problem which gave rise to the differential equation.

Hence, we may conclude, that *every differential equation of the n^{th} order between two variables has an integral, which, in its most general form, must contain n arbitrary constants.*

It is evident, that if we had differentiated the function $\phi(x, y, a, b \dots n)$ only $(n-1)$ times, we should on the whole have obtained n equations, from which we might eliminate $(n-1)$ constants, but no more; and since we might eliminate which $(n-1)$ of the constants we pleased, we should be able to obtain n different results, viz:

$$\left. \begin{array}{l} \psi \left(x, y, \frac{dy}{dx} \dots \frac{d^{n-1}y}{dx^{n-1}}, a \right) \\ f \left(x, y, \frac{dy}{dx} \dots \frac{d^{n-1}y}{dx^{n-1}}, b \right) \\ \dots \dots \dots \end{array} \right\} \dots (A),$$

from either of which the differential equation

$$F \left(x, y, \frac{dy}{dx} \dots \frac{d^n y}{dx^n} \right) \dots (B)$$

might be obtained by one differentiation, and the elimination of the one constant.

Hence, the n differential equations (A) are called the *first integrals* of the equation (B), and we conclude that *every differential equation of the n^{th} order, has n first integrals, which are differential equations of the $(n-1)^{\text{th}}$ order, and each contain one arbitrary constant.*

Although the above reasoning makes it appear that a differential equation of the n^{th} order *might* be formed from a function $\phi(x, y, a, b \dots n)$, or from the n differential equations (A), the following demonstrations are useful, inasmuch as they establish, more directly, that such is the constitution of every differential equation of the n^{th} order between two variables.

1st. *To shew that there is always an integral involving n arbitrary constants.*

Suppose $y = fx = X$ be the primitive, if there exist one of the differential equation; then, representing the differential coefficients $\frac{dX}{dx}, \frac{d^2X}{dx^2}, \dots \frac{d^nX}{dx^n}$ by $X', X'' \dots X^{(n)}$, we have, by Taylor's theorem,

$$\begin{aligned} f(x+h) = X + X'h + \frac{X''h^2}{1.2} + \frac{X'''h^3}{1.2.3} + \dots \\ \dots + \frac{X^{(n-1)}h^{n-1}}{1.2.3 \dots (n-1)} \dots \end{aligned}$$

and supposing a some value of x which does not cause a

failure of Taylor's theorem, we should have, representing the particular values of $X, X' \dots$ by $A, A' \dots$

$$f(a+h) = A + A'h + \frac{A''h^2}{1.2} + \frac{A'''h^3}{1.2.3} + \dots$$

$$\dots + \frac{A^{(n-1)}h^{n-1}}{1.2.3\dots(n-1)} \dots$$

or making the arbitrary quantity $h = (x - a)$

$$fx = A + A'(x-a) + \frac{A''(x-a)^2}{1.2} + \frac{A'''(x-a)^3}{1.2.3} \dots$$

$$\dots + \frac{A^{(n-1)}(x-a)^{n-1}}{1.2\dots(n-1)} + \text{terms} = K.$$

Now if our differential equation be of the n^{th} order, all the terms $= K$ might be determined, provided the value of the preceding n terms were known, because, barring the algebraic difficulties of the calculations, $X^{(n)}, X^{(n+1)} \dots$ and, consequently, $A^{(n)}, A^{(n+1)} \dots$ might be found in terms of the differential coefficients of orders inferior to the n^{th} , while the previous n quantities $A, A' \dots A^{(n-1)}$ remain arbitrary.

Hence there always exists some series, such as the above, which satisfies the differential equation; and this series, in its most general form, contains n arbitrary quantities, which, being functions of a , are constants.

2nd. *To shew that there are n integrals, each a differential equation of the $(n-1)^{\text{th}}$ order, and involving an arbitrary constant.*

Take the expansion

$$f(x+h) = X + X'h + \frac{X''h^2}{1.2} + \frac{X'''h^3}{1.2.3} \dots$$

$\therefore$ differentiating with respect to x ,

$$f'(x+h) = X' + X''h + \frac{X'''h^2}{1.2} \dots$$

$$f''(x+h) = X'' + X'''h + \dots$$

or, if $h = -x$

$$f0 = X - X'x + \frac{X''x^2}{1.2} - \frac{X'''x^3}{1.2.3} \dots$$

$$f'0 = X' - X''x + \frac{X'''x^2}{1.2} \dots$$

$$f''0 = X'' - X'''x + \dots$$

hence, if our equation were of the third order, we should be able to find these three series in terms of X and the differential coefficients X' , X'' , provided we knew the three constant values of X , X' , X'' , corresponding to $x = 0$ in them, (that is, the values of $f0, f'0, f''0$), and these are the three integrals of the second order, each involving an arbitrary constant.

Similarly, if we wrote down n consecutive series instead of three, we should shew that the theorem is true for a differential equation of the n^{th} order.

CHAPTER II.

ON EQUATIONS OF THE FIRST ORDER AND DEGREE.

UNDER this head we propose to investigate all such theorems as apply exclusively to the above class of equations.

SECT. 1. *Equations which are soluble merely by simple analytical artifices without reference to any of the theorems of differential equations.*

It is evident that if an equation can be put in the form

$$Xdx + Ydy = 0;$$

when X and Y are functions of x and y respectively, the integral is immediately found by integration, and we have

$$\int Xdx + \int Ydy = c,$$

c being the arbitrary constant.

Ex. Suppose we have the equation

$$\frac{dx}{\sqrt{1+x^2}} + adx + 2bydy = 0,$$

$$\therefore \int \frac{dx}{\sqrt{1+x^2}} + \int adx + \int 2bydy = c,$$

$$\text{or, } \log_e (x + \sqrt{1+x^2}) + ax + by^2 = c.$$

Equations of the form $Xdy + Ydx = 0$, are reduced to the last case by dividing by XY , when we shall have

$$\frac{dy}{Y} + \frac{dx}{X} = 0,$$

which may be integrated; or (if X_1 and Y_1 be functions of

x and y respectively, but different from X and Y) we may separate the variables in the equation

$$XYdx + X_1Y_1dy = 0,$$

by dividing by X_1Y , which gives us

$$\frac{X}{X_1}dx + \frac{Y_1}{Y}dy = 0,$$

an equation which may be integrated in the same manner as the first form.

A differential equation frequently gives rise to an integral of a transcendental form, which may sometimes be reduced to an algebraic form when the transcendental functions of x and y are both of the same sort.

Ex. 1. Suppose
$$\frac{dx}{1+x^2} + \frac{dy}{1+y^2} = 0,$$

$$\therefore \text{integrating } \tan^{-1}x + \tan^{-1}y = c,$$

$$\text{then suppose } \tan c = a,$$

$$\therefore \tan^{-1}x + \tan^{-1}y = \tan^{-1}a;$$

$$\therefore a = \frac{x+y}{1-xy}$$

is the algebraic integral required.

Ex. 2.
$$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} = 0,$$

$$\therefore \sin^{-1}x + \sin^{-1}y = c,$$

$$= \sin^{-1}a;$$

$$\therefore a = y\sqrt{1-x^2} + x\sqrt{1-y^2}$$

is the algebraic integral.

Ex. 3.
$$\frac{dx}{1-x^2} + \frac{ydy}{x(1+y^2)} = 0,$$

$$\therefore \frac{xdx}{1-x^2} + \frac{ydy}{1+y^2} = 0;$$

$$\therefore \log. (1+y^2) - \log. (1-x^2) = c = \log. a \text{ suppose};$$

$$\therefore a = \frac{1 + y^2}{1 - x^2}$$

is the algebraic integral.

Ex. 4. $adx = ydy - xdy;$

subtract ady from both sides,

$$\therefore a(dx - dy) = dy(y - a - x),$$

$$\frac{a(dx - dy)}{a + (x - y)} = -dy,$$

$$\frac{a(dx - dy)}{a + (x - y)} + dy = 0,$$

$$a \log_e (a + x - y) + y = \log_e c;$$

$$\therefore e^y \times (a + x - y)^a = c.$$

SECT. 2. *On the Equations which satisfy the criterion of integrability.*

There is an easy test, whereby we may determine whether a differential equation

$$Mdx + Ndy = 0$$

is such as has directly resulted from the differentiation of any function of x and y , and this is of much importance as we are enabled, when such is the case, to effect its immediate integration.

The reason why all equations are not immediately integrable, is, that many of them result from the elimination of constants, by differentiation and other operations, during which some common factor has been omitted; and, therefore, the equation no longer remains the *complete differential* of the primitive which belongs to it.

If $Mdx + Ndy = 0$ is a complete differential, then we shall have

$$\frac{dM}{dy} = \frac{dN}{dx}.$$

For suppose $Mdx + Ndy = 0$ to be the complete differ-

ential of the equation $u = c$, where u is some function of x and y , then we should have

$$\frac{du}{dx} = M, \text{ and } \frac{du}{dy} = N,$$

$$\therefore \frac{dM}{dy} = \frac{d^2u}{dx dy} \text{ and } \frac{dN}{dx} = \frac{d^2u}{dy dx};$$

but, by the principles of differential calculus,

$$\frac{d^2u}{dx dy} = \frac{d^2u}{dy dx}.$$

Therefore, if the equation has resulted directly from the differentiation of any function u , we must have

$$\frac{dM}{dy} = \frac{dN}{dx}.$$

This equation is called the criterion of integrability, which it is in one sense of the word, for when a differential equation does satisfy this criterion, it is immediately integrable by a very direct process, which we shall presently investigate; but, on the other hand, many equations which do not satisfy it may be integrated by other means.

It is evident, that equations in which the variables are separated, satisfy this condition; for the equation being of the form

$$Xdx + Ydy = 0,$$

$$\text{we have } \frac{dX}{dy} = 0, \frac{dY}{dx} = 0,$$

$$\therefore \frac{dX}{dy} = \frac{dY}{dx}, \text{ which is the condition required.}$$

To integrate an equation which satisfies the criterion of integrability.

Let $Mdx + Ndy = 0$ be the equation, and suppose $u = c$ to be its primitive, then M is the differential coefficient of u with respect to x alone, and N with respect to y alone; therefore we may integrate the term Mdx with respect to x only, remembering that since y is considered constant in

this integration, instead of adding a mere constant of integration, we must suppose a function of $y = Y$ added.

Therefore, if $\int Mdx + Y$ is the integral, N being the whole differential coefficient with respect to y , we have

$$\begin{aligned} N &= \frac{dfMdx}{dy} + \frac{dY}{dy}, \\ \therefore \frac{dY}{dy} &= \left(N - \frac{dfMdx}{dy} \right), \\ \therefore Y &= \int \left(N - \frac{dfMdx}{dy} \right) dy \end{aligned}$$

is determined.

Hence the complete integral is

$$\int Mdx + \int \left(N - \frac{dfMdx}{dy} \right) dy = c.$$

It is evident, that when the given equation is a perfect differential, the quantity $N - \frac{dfMdx}{dy}$, differentiated with respect to x , gives

$$\frac{dN}{dx} - \frac{d}{dx} \frac{dfMdx}{dy} = \frac{dN}{dx} - \frac{dM}{dy},$$

which must $= 0$, otherwise the criterion would not be fulfilled; therefore, Y is a function not containing x .

Ex. Let $(2axy - y^3) dx + (ax^2 - 3xy^2) dy = 0$,

$$\text{here } \frac{dM}{dy} = 2ax - 3y^2 = \frac{dN}{dx};$$

therefore the criterion of integrability is satisfied.

To find the integral, we have

$$\begin{aligned} \int Mdx &= ax^2y - y^3x, \\ \therefore \frac{dfMdx}{dy} &= ax^2 - 3y^2x = N \text{ in this case,} \\ \therefore \int \left(N - \frac{dfMdx}{dy} \right) dy &= \text{constant}; \end{aligned}$$

and, therefore, the integral of the given equation is

$$ax^2y - y^3x = c.$$

When the criterion of integrability is satisfied, the method of integration may be modified, so as to become more convenient in practice than the above; for if $Mdx + Ndy = 0$ be the differential equation, we have

$$\begin{aligned}\text{the integral} &= \int Mdx + Y, \\ &\text{or} = \int Ndy + X,\end{aligned}$$

where Y cannot contain x , and X cannot contain y ; therefore, if we find $\int Mdx$ and $\int Ndy$, the functions Y and X will be determined.

An example will make this easily understood. Suppose we have

$$\left(\frac{ax}{\sqrt{y^2 + x^2}} + \frac{y}{y^2 + x^2} \right) dx + \left(\frac{ay}{\sqrt{y^2 + x^2}} - \frac{x}{y^2 + x^2} + 3by^2 \right) dy = 0,$$

it easily appears that the criterion of integrability is satisfied.

Then, integral

$$= \int Mdx + Y = a\sqrt{y^2 + x^2} - \tan^{-1} \frac{y}{x} + Y,$$

$$= \int Ndy + X = a\sqrt{y^2 + x^2} - \tan^{-1} \frac{y}{x} + by^3 + X,$$

and these are identical;

$$\therefore X = 0, Y = by^3;$$

therefore, the integral is $a\sqrt{y^2 + x^2} - \tan^{-1} \frac{y}{x} + by^3 = c$.

SECT. 3. *On factors which render differential equations integrable.*

We have observed above, that many differential equations fail to satisfy the criterion of integrability, in consequence of the omission of some common factor, which would exist if the equations had been obtained directly from their primitives by differentiation.

Thus, the equation

$$\left(\frac{a}{x} - \frac{cx^m}{y^p} \right) dx + \frac{b}{y} dy = 0$$

evidently does not satisfy the criterion, but if we multiply by $x^a y^b$, it becomes

$$(ax^{a-1}y^b - cx^{m+a}) dx + by^{b-1}x^a dy = 0,$$

$$\text{and } \frac{dM}{dy} = abx^{a-1}y^{b-1}; \quad \frac{dN}{dx} = aby^{b-1}x^{a-1};$$

therefore, the criterion is now satisfied, and the equation becomes immediately integrable.

When the integrating factor is of so simple a form as the above, it may generally be discovered by inspection, but there are some general theorems respecting these factors which lead to their discovery in other more complicated cases, and which we shall proceed to investigate.

There exist always an infinite number of factors, which will render any given differential equation of the first order integrable.

1st. There will always exist one factor, &c.

For, let the equation be put in the form

$$\frac{dy}{dx} + f(x, y) = 0,$$

and since there must be some primitive belonging to it, suppose that primitive to be represented by $F(x, y) = a$; where a is a constant which has disappeared in the differentiation.

Then differentiating, and representing by $F'x$ the differential coefficient with respect to x , and $F'y$ that with respect to y , we have

$$F'x + F'y \frac{dy}{dx} = 0,$$

which is a complete differential; or, dividing by $F'y$, we have

$$\frac{dy}{dx} + \frac{F'x}{F'y} = 0,$$

which, as it arises from the same primitive, and is of the same form as the given equation, must be identical with it ;

$$\therefore \frac{dy}{dx} + \frac{F'x}{F'y} = 0 \text{ is identical with } \frac{dy}{dx} + f(x, y) = 0,$$

$$\therefore \left(\frac{dy}{dx} + f(x, y) \right) F'y \text{ identical with } \frac{dy}{dx} F'y + F'x = 0;$$

and the latter is a complete differential, therefore the former is so likewise; and since the equation always has a primitive, there always exists some quantity $F'y$, which will make it a perfect differential.*

2nd. There will be an infinite number of such factors. For if μ be the one factor which we have shewn always to exist,

$$\text{then } \mu \cdot \left(\frac{dy}{dx} + f(x, y) \right) \text{ is a complete differential,} \\ = du, \text{ suppose,}$$

$$\therefore \mu \cdot \phi u \left(\frac{dy}{dx} + f(x, y) \right) = \phi u du$$

is always a complete differential.

Hence $\mu\phi u$ is a factor, which renders the equation a perfect differential whatever the form of ϕu may be, and ϕu may have an infinite number of forms; and, therefore, there are an infinite number of factors which satisfy the conditions.

* By similar reasoning, it is shewn that there always exists an integrating factor for an equation of the n^{th} order. For, suppose the equation put in the form

$$\frac{d^n y}{dx^n} + \phi \left(x, y, \frac{dy}{dx} \dots \frac{d^{n-1}y}{dx^{n-1}} \right) = 0, \text{ or } \frac{d^n y}{dx^n} + U = 0,$$

and suppose its integral to be

$$f \left(x, y, \frac{dy}{dx} \dots \frac{d^{n-1}y}{dx^{n-1}} \right) = a, \text{ or } U_1 = a.$$

This differentiated, gives an equation of the form

$$u \frac{d^n y}{dx^n} + u_1 = 0;$$

(u and u_1 being functions of x, y , and the differential coefficients of orders inferior to the n^{th}).

Then, dividing by u ,

$$\frac{d^n y}{dx^n} + \frac{u_1}{u} = 0 \text{ is identical with } \frac{d^n y}{dx^n} + U = 0,$$

$$\therefore u \frac{d^n y}{dx^n} + u_1 = 0 \dots \dots \dots u \frac{d^n y}{dx^n} + uU = 0;$$

therefore, u represents the integrating factor which always exists.

COR. If the integral of a given differential equation is known, an infinite number of integrating factors may be determined; for suppose u the function of x and y , which is the integral, and μ the integrating factor,

$$\text{then } \mu M dx + \mu N dy = du$$

is identical with

$$\frac{du}{dx} dx + \frac{du}{dy} dy = du;$$

therefore, comparing these, the value of μ may be found, and since u is known, an infinite number of factors included in the form $\mu \phi u$ may be determined.

To investigate an equation from which the integrating factor may, in certain cases, be determined.

Suppose $M dx + N dy = 0$ the differential equation, and let μ be the factor required to make it a complete differential; then $\mu M dx + \mu N dy = 0$, being complete, must satisfy the criterion of integrability;

$$\begin{aligned} \therefore \frac{d(\mu M)}{dy} &= \frac{d(\mu N)}{dx}, \\ \text{or, } \mu \frac{dM}{dy} + M \frac{d\mu}{dy} &= \mu \frac{dN}{dx} + N \frac{d\mu}{dx}, \end{aligned}$$

which is the equation required.

The factor can be determined from the above, if μ is a function of x only.

For, then $\frac{d\mu}{dy} = 0$, and the equation becomes

$$\begin{aligned} \mu \frac{dM}{dy} &= \mu \frac{dN}{dx} + N \frac{d\mu}{dx}, \\ \therefore \mu \left(\frac{dM}{dy} - \frac{dN}{dx} \right) &= N \frac{d\mu}{dx}, \\ \therefore \frac{d\mu}{\mu} &= \frac{dx}{N} \cdot \left\{ \frac{dM}{dy} - \frac{dN}{dx} \right\}, \\ \therefore \log_e \mu &= \int \frac{dx}{N} \cdot \left\{ \frac{dM}{dy} - \frac{dN}{dx} \right\}, \end{aligned}$$

$$\therefore \mu = e^{\int \frac{dx}{N} \cdot \left\{ \frac{dM}{dy} - \frac{dN}{dx} \right\}}.$$

Similarly, if μ were a function of y alone, we should have

$$\mu = e^{\int \frac{dy}{M} \cdot \left\{ \frac{dN}{dx} - \frac{dM}{dy} \right\}}.$$

Ex. 1. The equation $dy + (Py - Q) dx = 0$, called the linear equation, in which P and Q are functions of x alone, forms a simple example of the case in which the integrating factor is a function of x only; for we find

$$\frac{dM}{dy} - \frac{dN}{dx} = P,$$

and $N = 1$; therefore, the integrating factor will be $e^{\int P dx}$.

Hence, multiplying the equation by it

$$e^{\int P dx} dy + e^{\int P dx} Py dx = e^{\int P dx} Q dx;$$

the first side is evidently the differential of $y \cdot e^{\int P dx}$, and the second is a function without y ;

$$\therefore y \cdot e^{\int P dx} = \{c + \int Q e^{\int P dx} dx\}$$

is the integral of the proposed equation, where c represents the whole constant arising from the integration of $P dx$, as well as of $Q e^{\int P dx} dx$.

$$\text{Ex. 2.} \quad \frac{y}{x^2} dx + \left(ay^n - \frac{2}{x} \right) dy = 0;$$

here we find the quantity

$$\int \frac{dy}{M} \cdot \left\{ \frac{dM}{dx} - \frac{dM}{dy} \right\} = \int \frac{x^2 dy}{y} \cdot \left\{ \frac{2}{x^2} - \frac{1}{x^2} \right\} = \int \frac{dy}{y} = \log. y,$$

$$\therefore \text{the factor} = e^{\int \frac{dy}{y}} = y;$$

then multiplying by this factor, the equation becomes

$$\frac{y^2}{x^2} dx + \left(ay^{n+1} - \frac{2y}{x} \right) dy = 0,$$

which satisfies the criterion of integrability, and gives the integral

$$\frac{ay^{n+2}}{n+2} = \frac{y^2}{x} + c.$$

On the inverse problem of factors.

The above investigations have for their object the finding such factors as may render particular classes of equations integrable, while the *inverse problem* consists in finding the nature of such differential equations as may be made integrable by a factor of an assumed form. Little practical good has as yet resulted from the labours of analysts on this point, and we shall not therefore devote much time to it.

Any equation being represented by $Mdx + Ndy = 0$; to find the nature of such equations as may be made integrable by a factor of the form $\frac{1}{Mx + Ny}$.

Since the equation $\frac{Mdx + Ndy}{Mx + Ny} = 0$

is to be an exact differential, we must have

$$\frac{d}{dy} \left(\frac{M}{Mx + Ny} \right) = \frac{d}{dx} \left(\frac{N}{Mx + Ny} \right);$$

or, assuming $N = rM$, when r is a function of x and y , the equation becomes

$$x \frac{dr}{dx} + y \frac{dr}{dy} = 0;$$

or, representing the partial differential coefficients of r , by m and n ,

$$mx + ny = 0 \dots (1),$$

$$\text{and } dr = m dx + n dy,$$

$$\therefore n = \frac{dr - m dx}{dy};$$

therefore, substituting this value of n in (1), we have

$$mx + y \cdot \frac{dr - m dx}{dy} = 0,$$

$$\begin{aligned} \text{whence } dr &= my \cdot \left(\frac{y dx - x dy}{y^2} \right) \\ &= my d \left(\frac{x}{y} \right); \end{aligned}$$

and this must be a complete differential.

$$\therefore dr = f\left(\frac{x}{y}\right) d\left(\frac{x}{y}\right),$$

$$\therefore r = F\left(\frac{x}{y}\right),$$

$$\text{that is, } \frac{N}{M} = F\left(\frac{x}{y}\right);$$

or, N and M must be of such a nature that their quotient may be of no dimensions in x and y ; therefore, the functions N and M must be homogeneous with respect to x and y .

To assign the form of the indeterminate functions of x and X_1 , in the equation $Xydx + X_1dx + ydy = 0$, in order that it may become integrable when multiplied by the factor $\frac{1}{(y + fx)^n}$.

Represent the factor by μ ; then the general equation for finding the integrating factor, is

$$M \frac{d\mu}{dy} - N \frac{d\mu}{dx} = \left(\frac{dN}{dx} - \frac{dM}{dy}\right) \mu;$$

and, in this case, it becomes

$$(Xy + X_1) \frac{d\mu}{dy} - y \frac{d\mu}{dx} = -X\mu;$$

$$\text{but } \frac{d\mu}{dy} = -\frac{n}{(y + fx)^{(n+1)}}, \text{ and } \frac{d\mu}{dx} = -\frac{nf'x}{(y + fx)^{(n+1)}};$$

then, substituting these values in the last equation, it gives us

$$\{(n-1)X - nf'x\} y + nX' - Xfx = 0,$$

and since this must hold, independently of the value of y , we have for determining X and X_1 , the two equations

$$\left. \begin{aligned} (n-1)X - nf'x &= 0 \\ nX' - Xfx &= 0 \end{aligned} \right\}.$$

SECT. 4. On homogeneous Equations.

A differential equation is said to be homogeneous when the sum of the indices of the variables in each term is the same.

Thus, if in the equation

$$x^p y^q dx + x^r y^s dy = 0,$$

$$p + q = r + s,$$

the equation is homogeneous; or, if the equation be

$$\frac{x^p}{y^q} dx + \frac{x^r}{y^s} dy = 0,$$

and $p - q = r - s$, the equation is homogeneous.

To find generally the factor which will render a homogeneous equation integrable.

Let $Mdx + Ndy = 0$ be the differential equation, and suppose $u = f(x, y)$ the primitive, and μ the integrating factor. Then, because the differential is homogeneous, the primitive must be so also, when the arbitrary constant is supposed to vanish, and we may represent its dimensions by m ; again, u being homogeneous, may be put in the form

$$F\left(\frac{x}{y}\right) = a;$$

and, therefore, if x becomes $(1 + \alpha)x$, y must become $(1 + \alpha)y$, and u will become $(1 + \alpha)^m u$; therefore, by the expansion of a function of two variables, we have

$$(1 + \alpha)^m u = u + \frac{du}{dx} \alpha x + \frac{du}{dy} \alpha y,$$

$$+ \text{terms of } \alpha^2, \alpha^3, \dots$$

$$\therefore u + m\alpha u + \dots = u + M_{\mu}\alpha x + N_{\mu}\alpha y + \dots$$

Therefore, equating coefficients of like powers of α , since that quantity is indeterminate,

$$\text{we have } m\mu u = \mu Mx + \mu Ny;$$

$$\text{but } du = \mu Mdx + \mu Ndy,$$

$$\therefore \frac{du}{mu} = \frac{Mdx + Ndy}{Mx + Ny}.$$

Now the first side is always a complete differential, and therefore the second side is so also, and hence we see that $\frac{1}{Mx + Ny}$ is the required factor.

COR. From the above, it appears that if the equation $Pdx + Qdy = 0$ be homogeneous, and also a complete differential, its integral may be immediately obtained.

For, if u be the integral, and n the dimensions of the differential equation in x and y ,

$$(n + 1)u = Px + Qy;$$

$$\therefore \frac{Px + Qy}{n + 1} = c$$

is the complete integral.

Ex. The equation $(x^2y + y^3) dx - 2xy^2dy = 0$, has for the integrating factor

$$\frac{1}{x^3y + xy^3 - 2xy^3} = \frac{1}{x^3y - xy^3};$$

therefore, multiplying by this factor, it becomes

$$\frac{1}{x} \cdot \frac{x^2 + y^3}{x^2 - y^3} dx - \frac{2y}{x^2 - y^3} dy = 0;$$

and we shall easily perceive that this equation satisfies the criterion of integrability,

$$\begin{aligned} \text{for } \frac{dM}{dy} &= \frac{1}{x} \cdot \frac{(x^2 - y^3) 2y + (x^2 + y^3) 2y}{(x^2 - y^3)^2}, \\ &= \frac{1}{x} \cdot \frac{4x^2y}{(x^2 - y^3)^2} = \frac{4xy}{(x^2 - y^3)^2}, \\ &= \frac{dN}{dx}, \therefore N = -\frac{2y}{x^2 - y^3}; \end{aligned}$$

therefore the equation is integrable by the method investigated in page 10. †

There is, however, a general method of solving homogeneous equations without investigating the integrating factor.

Suppose m be the sum of the indices of the variables in each term of the given equation; then, dividing the whole by x^m or by y^m , it will take the form

$$f\left(\frac{y}{x}\right) dx + \phi\left(\frac{y}{x}\right) dy = 0,$$

$$\text{or } f\left(\frac{x}{y}\right) dx + \phi\left(\frac{x}{y}\right) dy = 0,$$

respectively.

Supposing the former case assume

$$\frac{y}{x} = z,$$

$$\therefore y = xz \text{ and } dy = xdz + zdx;$$

also, dividing the whole equation by $\phi\left(\frac{y}{x}\right)$, we shall put it in the form

$$dy + F(z)dx = 0,$$

and substituting the value of dy , we have

$$xdz + zdx + F(z)dx = 0,$$

$$\therefore \frac{dz}{z + F(z)} + \frac{dx}{x} = 0;$$

and, in this equation, the variables being separated, we may integrate, after which we must substitute the value $\frac{y}{x}$ for z .

Similarly, if we had divided by y^m , the equation would become soluble by putting $\frac{x}{y} = z$.

Inspection of the given equation, and a little practice, will generally teach us which plan is to be preferred in any particular case.

$$\text{Ex. 1. } xdy - (y + \sqrt{x^2 + y^2}) dx = 0,$$

$$\therefore dy - \left\{ \frac{y}{x} + \left(1 + \frac{y^2}{x^2} \right)^{\frac{1}{2}} \right\} dx = 0;$$

$$\text{then putting } \frac{y}{x} = z, y = xz,$$

$$\therefore dy = xdz + zdx;$$

and making these substitutions in the equation, it gives

$$xdx - \sqrt{1 + z^2} dx = 0,$$

$$\therefore \frac{dz}{\sqrt{1 + z^2}} - \frac{dx}{x} = 0,$$

$$\therefore \log_e (z + \sqrt{1+z^2}) - \log_e x = \log_e c$$

$$\frac{z + \sqrt{1+z^2}}{x} = c,$$

which, putting $\frac{y}{x}$ for z , gives

$$\frac{y + \sqrt{x^2 + y^2}}{x^2} = c.$$

Ex. 2. $xdy - \left\{ y + y \log_e \left(\frac{y}{x} \right) \right\} dx = 0,$

put $\frac{y}{x} = z$, and the equation will become

$$\frac{dx}{x} = \frac{dz}{z \cdot \log_e z},$$

$$\therefore \log_e x + \log_e c = \log_e (\log_e z),$$

$$\therefore cx = \log_e z,$$

$$\therefore z = e^{cx},$$

$$y = xe^{cx}.$$

We frequently meet with equations which, although not homogeneous, are easily made so by proper substitutions.

Ex. Let the equation be

$$(ax + by + c) dy + (mx + ny + p) dx = 0,$$

which is not homogeneous,

$$\text{assume } ax + by + c = z, \therefore adz + bdy = dz,$$

$$. \quad . \quad mx + ny + p = t, \therefore mdx + ndy = dt;$$

hence, eliminating first dy and then dx , we have, from these two equations,

$$dx = \frac{ndz - bdt}{na - mb},$$

$$dy = \frac{adt - mdz}{na - mb};$$

therefore, omitting the common factor $\frac{1}{na - mb}$, our equation gives us

$$z(adt - mdz) + t(ndz - bdt) = 0,$$

$$\text{or } (az - bt) dt + (nt - mz) dz = 0,$$

which is homogeneous, and may be solved by the general method.

The above process fails when $na = mb$, for then we see that the values of dx and dy found above would become infinite, which indicates that the course we have pursued is not applicable to the case in question; and, although the factor $\frac{1}{na - mb}$ is common to the whole equation, and therefore may be left out, still if we proceed in this manner with the equation in which $na = mb$, our final result, instead of being a function of x and y , would be the identical equation, constant = c .

To solve the equation in this case, we have to observe that

$$m = \frac{na}{b},$$

$$\therefore \left(\frac{na}{b} x + ny + p \right) dx + (ax + by + c) dy = 0,$$

$$\text{or } bcdy + bpdx + (ax + by)(bdy + ndx) = 0;$$

$$\text{then, assuming } ax + by = z,$$

$$adx + bdy = dz, \therefore dy = \frac{dz - adx}{b};$$

and substituting, we have

$$cdz - cadx + bpdx + z \cdot (dz - adx + ndx) = 0,$$

$$\text{or } dx \{ (n - a)z + (bp - ca) \} + (c + z) dz = 0,$$

$$\therefore dx + \frac{(c + z) dz}{(n - a)z + bp - ca} = 0,$$

which is an equation in which the variables are separated, and therefore it may be integrated.

The solution of the equation $dy + (ay^2 + bx^m) dx = 0$, may, under certain conditions, be reduced to the solution of a homogeneous equation, or of one in which the variables are immediately separable.

If $m = 0$, it becomes

$$dx + \frac{dy}{ay^2 + b} = 0,$$

in which the variables are separated.

But if m be not $= 0$, assume

$$y = \frac{z}{x^2} + \frac{1}{ax},$$

$$\therefore dy = \frac{xdz - 2zdx}{x^3} - \frac{dx}{ax^2},$$

and substituting these values in the given equation, it becomes

$$x^2 dz + az^2 dx + bx^{m+4} dx = 0.$$

If in this $m = -2$, it is homogeneous; and if $m = -4$, the variables may be immediately separated by dividing the whole equation by $x^2 (az^2 + b)$.

If, however, m have not either of the above values, we must again transform our equation, by assuming

$$\left. \begin{array}{l} z = \frac{1}{t} \text{ and } x^{m+3} = u, \\ \text{and putting } n \text{ for } -\frac{b}{m+3}, \\ \quad \quad \quad a' \text{ for } \frac{b}{m+4}, \\ \quad \quad \quad \text{and } b' \text{ for } -\frac{a}{m+3}, \end{array} \right\} (a),$$

our transformed equation becomes

$$dt + (a't^2 + b'u^n) du = 0;$$

and this being similar to the original equation, may be integrated if $n = -2$, or $n = -4$.

If n be not $= -2$, nor $= -4$, by continually repeating the same transformation as the last, the equation may successively be reduced to a series of equations of the same form as the given one, and in which the exponent of the variable becomes successively equal to

$$-\frac{m+4}{m+3}, -\frac{3m+8}{2m+5}, -\frac{5m+12}{3m+7}, -\frac{7m+16}{4m+9} \dots$$

We can only arrive at the solution of the equation by the above method when some one of these quantities is either $= 0$, or $= -2$, or $= -4$, that is, when m is a number of the form $\frac{-4p}{2p-1}$, p being any positive integer.

If the transformation $y = \frac{1}{z}$, $x^{m+1} = u$ had been made in the given equation at once, the same process would shew that the integration could only be effected if m were of the form $\frac{-4p}{2p+1}$.

Hence, *first*, if m be of the form $\frac{-4p}{2p-1}$ we must proceed as above; *secondly*, if m be of the form $\frac{-4p}{2p+1}$ we must commence at once with the substitutions (a), and proceed as there directed.

The above is usually called *Riccati's* equation, because its properties were first investigated by that mathematician.

SECT. 5. *On Equations which are reducible to the form of the linear equation of the first order.*

An equation is called linear when it involves only the first power of y and its differential coefficients, the other quantities being functions of x , and constants; when the first differential coefficient only is involved, the equation is called the linear equation of the first order.

The general form of this equation is

$$dy + Pydx = Qdx,$$

where P and Q are functions not containing y ; its solution has already been effected in page (15) by means of the integrating factor, which we found to be $e^{\int Pdx}$; and this, perhaps, is the best process that can be employed. There is, however, another which it may not be amiss to give here, as it is also applicable to linear equations of the second order.

In the equation $dy + Pydx = Qdx$,

assume $y = uz$,

$$\therefore dy = u dz + z du;$$

therefore, substituting, we have

$$udz + zdu + Pzudx = Qdx;$$

and since there are two indeterminate quantities, z and u , concerning which we have only made one supposition, we are at liberty to make another; and, therefore, we may assume them such, that the whole coefficient of z in the above shall equal nothing. This supposition gives us the two equations

$$Pudx + du = 0,$$

$$udz = Qdx.$$

From the first we have $u = e^{-\int Pdx}$,

and from the second $z = c + \int e^{\int Pdx} Qdx$;

$$\therefore y = e^{-\int Pdx} \cdot \{c + \int e^{\int Pdx} Qdx\};$$

which is the same result that was obtained in page 15.

In the following example of the linear equation we apply the method previously deduced, because it is generally the neatest in practice.

Ex. $(1 - x^2) dy + xydx = adx,$

$$\text{gives } dy + \frac{x}{1-x^2} ydx = \frac{adx}{1-x^2}$$

$$\text{Here } \int Pdx = \int \frac{xdx}{1-x^2} = \log_e \frac{1}{\sqrt{1-x^2}},$$

therefore the integrating factor $= e^{\int Pdx} = \frac{1}{\sqrt{1-x^2}}$, and the equation, when multiplied by the factor, gives

$$\frac{dy}{\sqrt{1-x^2}} + \frac{x}{(1-x^2)^{\frac{3}{2}}} ydx = \frac{adx}{(1-x^2)^{\frac{3}{2}}},$$

$$\therefore \frac{y}{\sqrt{1-x^2}} = \frac{ax}{\sqrt{1-x^2}} + c,$$

$$\text{or } y = ax + c \sqrt{1-x^2}.$$

E

We not unfrequently meet with equations of the form

$$dx + Rxdy = Sdy,$$

where R and S are functions of y not containing x , and this is a linear equation only with x in place of y , and vice versa; and although x has hitherto been considered the independent variable, still as the differential coefficients in the above do not rise above the first order, we may, without making any alteration in the equation, consider y as the independent variable, and proceed with the integration precisely as in the last case, when we shall find for the integral

$$x = e^{-\int Rdy} \{c + \int e^{\int Rdy} Sdy\}.$$

There is an important class of differential equations which are reducible to the form of the linear equation by a very simple substitution; these are included in the form

$$\left(y^n dy + Py^{n+1} dx = Qdx, \right)$$

where the dimensions of y in the second term is greater by unity than in the first; to reduce the equation to the proposed form, assume

$$\begin{aligned} y^{n+1} &= u, \\ \therefore (n+1) y^n dy &= du, \\ \therefore y^n dy &= \frac{du}{(n+1)}. \end{aligned}$$

Substituting this value in the equation, and multiplying by $(n+1)$, we have

$$du + (n+1) P u dx = (n+1) Q dx,$$

which is a linear equation of the first order between u and x .

It is scarce necessary to observe that the equation

$$x^n dx + Rx^{n+1} dy = Sdy$$

may be solved by similar means.

Ex. 1. $y^3 dy + y^3 dx = ax dx,$

$$\text{assume } y^3 = u,$$

$$\therefore 3y^2 dy = du;$$

therefore, substituting, we have

$$du + 3u dx = 3ax dx,$$

which is a linear equation, the integral of which is

$$3xu = 3ax^3 + 3c,$$

$$\text{or } y^3 = ax^2 + \frac{c}{x}.$$

The method of solution applies equally when the indices of y are negative, provided the index in the second term be greater by unity than that in the first.

Ex. 2. $\frac{dy}{y^3} + \frac{dx}{y^2} = xdx,$

$$\text{assume } u = \frac{1}{y^2};$$

$$\therefore du = -\frac{2dy}{y^3};$$

therefore, substituting $du - 2udx = -2xdx$, which is linear, and may be integrated.

Similarly, when the indices are fractional.

Ex. 3. $\frac{dy}{\sqrt{y}} + \frac{x\sqrt{y} dx}{1-x^2} = xdx,$

$$\text{assume } u = \sqrt{y},$$

$$\therefore du = \frac{dy}{2\sqrt{y}};$$

$$\therefore du + \frac{xu}{2(1-x^2)} dx = \frac{xdx}{2}.$$

This is a linear equation, and integrating it, we find

$$\sqrt{y} = c \sqrt{1-x^2} - \frac{1-x^2}{3}.$$

CHAPTER III.

ON SINGULAR OR PARTICULAR SOLUTIONS.

THE present chapter is devoted to this branch of the subject, because equations of a degree superior to the first generally admit of *singular* solutions, and therefore some knowledge of these is necessary to enable the student to proceed to the integration of such equations.

To explain the nature of singular solutions, and shew how to obtain them from the complete primitive.

We have hitherto taken for granted, whenever a function $f(x,y)$ satisfies a given differential equation of the first order, that on adding a constant of integration, it is to be considered as the *complete integral*. This, however, is not always the case, as we sometimes meet with a function of x and y , the differential of which coincides with the proposed equation, while the function itself neither coincides with, nor is contained in the complete or general integral.

Such a function then, inasmuch as it satisfies the differential equation, must be considered a *solution* of it, but as it is not the species of solution which is generally the object of investigation, or even one of which all differential equations are susceptible, it is called a *particular* or *singular solution*.

To shew how this may happen, let us suppose the complete primitive to have been $V = 0$, where V is a function of x and y containing a constant c ; then the differential equation will be

$$\frac{dV}{dx} dx + \frac{dV}{dy} dy = 0, \dots (a);$$

or, if we suppose for an instant that c were variable also, the differential equation would be

$$\frac{dV}{dx} dx + \frac{dV}{dy} dy + \frac{dV}{dc} dc = 0;$$

and any supposition that would make the third term of this last vanish, would make it coincide with the equation (a);

and, therefore, if by equating $\frac{dV}{dc}$ to zero, we find a value of c in terms of x and y , this value of c substituted in $V = 0$, will give us an equation between x and y , $V_1 = 0$, which, although not contained in the form $V = 0$, will give the same differential equation.

Hence $V_1 = 0$, is a singular solution of equation (a), and we see that generally if the complete primitive be such that the assumption $\frac{dV}{dc} = 0$ would give us a value of c in terms of x and y , the differential equation will admit of a singular solution.

Similarly, if $V = 0$ be a function containing two or more constants, the elimination of each of these constants, according to the above conditions, will give singular solutions.

Ex. Take the equation

$$\sqrt{x^2 + y^2 - a^2} = y + c$$

(which is the complete primitive of the differential equation

$$x dx + (y - \sqrt{x^2 + y^2 - a^2}) dy = 0).$$

Squaring both sides, we have the equation

$$x^2 - 2cy - c^2 - a^2 = 0;$$

therefore, differentiating with respect to c , we have

$$\frac{dV}{dc} = -2y - 2c,$$

and equating this to nothing, it gives us

$$c = -y;$$

therefore, substituting this value for c in the complete primitive, we shall have the singular solution, which is

$$x^2 + y^2 - a^2 = 0.$$

If $V = 0$ be a singular solution of a differential equation, and if the values of $\frac{dc}{dx}$ and $\frac{dc}{dy}$ be found from the complete primitive, then the relation expressed by $V = 0$ will make $\frac{dc}{dx}$ and $\frac{dc}{dy}$ infinite.

For we have, from equation $V = 0$,

$$\frac{dV}{dx} + \frac{dV}{dc} \frac{dc}{dx} = 0, \text{ and } \frac{dV}{dy} + \frac{dV}{dc} \frac{dc}{dy} = 0;$$

then neither $\frac{dV}{dx}$ nor $\frac{dV}{dy}$ are *generally* equal to zero; therefore, neither $\frac{dV}{dc} \frac{dc}{dx}$ nor $\frac{dV}{dc} \frac{dc}{dy}$ can equal zero.

But if $V = 0$ is a singular solution, $\frac{dV}{dc}$ is equal to zero; therefore the other factors must equal infinity.

$$\therefore \frac{dc}{dx} = \text{infinity, and } \frac{dc}{dy} = \text{infinity.}$$

COR. Hence we may sometimes find singular solutions from the complete primitive by investigating the conditions under which $\frac{dc}{dx}$ and $\frac{dc}{dy}$ become infinite.

Ex. Let $c^2 + 2cy = x^2 + y^2$ be the complete primitive;

$$\therefore c + y = \pm \sqrt{x^2 + 2y^2},$$

$$\therefore \frac{dc}{dy} + 1 = \pm \frac{2y}{\sqrt{x^2 + 2y^2}}.$$

Now this will equal infinity if $x^2 + 2y^2 = 0$; and $x^2 + 2y^2 = 0$ is precisely the equation we should have found by eliminating c by the first process, that is, it is a singular solution.

Having given a relation $y = X$, which satisfies a proposed differential equation of the first order, to determine whether or not it is a singular solution.

Suppose $y = (X + z)$ to be the complete solution where z is a quantity containing an arbitrary constant c , and may contain x . Now z may vanish, and therefore $y = X$ satisfy the given differential equation on two suppositions; *first*, by supposing c to have some particular constant value, in which case $y = X$ is merely a particular integral; and, *secondly*, by giving c some value in terms of x , in which case $y = X$ is a singular or particular solution.

Let the proposed differential equation be put in the form $\frac{dy}{dx} - p = 0$, where $p = f(x, y)$,

$$\text{then } \frac{dX}{dx} = f(x, X),$$

$$\text{and } \frac{d(X + z)}{dx}, \text{ or } \frac{dX}{dx} + \frac{dz}{dx} = f\{x, (X + z)\};$$

then, supposing this expanded by some algebraic means, we should have

$$\frac{dX}{dx} + \frac{dz}{dx} = f(x, X) + Pz^m + Qz^n + \dots$$

where P and Q are functions of x and the indices $m, n, \dots$ are in order of magnitude, m being the least;

$$\therefore \frac{dz}{dx} = Pz^m + Qz^n + \dots$$

Now, since z may vanish, it may also become so small that all terms after the first may be neglected, and this being the case,

$$z^{-m} dz = P dx,$$

$$\therefore \frac{z^{1-m}}{1-m} = c + \int P dx;$$

then, if $m = 1$, when z vanishes $c + \int P dx = \infty$, $\therefore c = \infty$,

if $m > 1$, $c + \int P dx = \infty$, $\therefore c = \infty$,

but if $m < 1$, $c + \int P dx = 0$;

$$\therefore c = -\int P dx;$$

therefore, in the two first cases, $X = 0$ is only a particular integral, because the supposition of z vanishing only gives $c = \infty$, but in the last case, the supposition of z vanishing gives $c = \text{function of } x$; therefore $y = X$ is a singular solution.

Hence, to find whether $y = X$ is a particular integral or a singular solution, put the differential equation in the form

$$\frac{dy}{dx} = f(x, y);$$

in this function $f(x, y)$ put $X + z$ for y , where z is indeterminate, and expand the resulting function $f\{x, (X + z)\}$ in a series ascending by powers of z (by some algebraic process); then, if the least power of z is equal or greater than unity $y = X$ is a particular integral; but if there be a power of z less than unity, $y = X$ is a singular solution.

COR. We may hence deduce a method of finding singular solutions from *differential* equations. For, since the first power of z in the development of $f\{x, (X + z)\} = \frac{dX}{dx} + \frac{dz}{dx}$ is fractional, *Taylor's* theorem would fail in the first term of the development of this quantity, which is $p + \delta p$ varying dependently upon a variation of y ; therefore $\frac{dp}{dy}$ must = infinity; therefore, having obtained the value of $\frac{dp}{dy}$ from the differential equation, the conditions that make this infinite will give us a singular solution, provided the result satisfies the differential equation.

Ex. The equation

$$4y = 4x \frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2$$

is satisfied by the integral $y = x^2$; to find whether or not this is a singular solution.

We have, from the equation,

$$\frac{dy}{dx} = p = 2(x \pm \sqrt{x^2 - y});$$

therefore, putting $x^2 - z$ for y , we get

$$p_1 = 2x \pm 2z^{\frac{1}{2}};$$

therefore, the first exponent of z being less than unity, $y = x^2$ is a singular solution.

We have in the above, taken z with a negative sign, merely to avoid the appearance of an impossible quantity; it is evident, since z is indeterminate, we are at liberty to consider it either positive or negative, at pleasure.

Ex. Given a solution $y = x$, which satisfies the equation $a^2 dy + (y^2 - xy - a^2) dx = 0$, to find whether or not it is a singular solution.

From the differential equation, we have

$$\frac{dy}{dx} = p = 1 + \frac{xy - y^2}{a^2};$$

therefore, putting $x + z$ for the value of y instead of x , we get

$$\begin{aligned} p_1 &= 1 + \frac{x \cdot (x + z) - (x + z)^2}{a^2}, \\ &= 1 - \frac{xz}{a^2} - \frac{z^2}{a^2} \dots \end{aligned}$$

which (involving no power of z less than the simple power) shews us that the solution $y = x$ is *not* a singular solution.

It is, therefore, a particular integral corresponding to an infinite value of the constant, and by means of it we may find the complete integral, as follows:

$$\text{we have } \frac{dy}{dx} = 1 + \frac{xy - y^2}{a^2};$$

$$\therefore \text{ assume } y = x + \frac{a^2}{v},$$

$$\therefore \frac{dy}{dx} = 1 - \frac{a^2}{v^2} \frac{dv}{dx};$$

then, substituting these values, we obtain the equation

$$\frac{a^2}{v^2} \frac{dv}{dx} = \frac{a^2}{v^2} + \frac{x}{v},$$

$$\frac{dv}{dx} = 1 + \frac{vx}{a^2};$$

this is a linear equation, which gives

$$v = e^{\frac{x^2}{2a^2}} \{c + \int e^{-\frac{x^2}{2a^2}} dx\},$$

$$\therefore y = x + \frac{a^2 e^{-\frac{x^2}{2a^2}}}{c + \int e^{-\frac{x^2}{2a^2}} dx}$$

is the general solution, which evidently becomes $y = x$, when $c = \infty$.

It is important to observe the method by which we have deduced the complete primitive from its particular value, corresponding to $c = \infty$; as, in many cases, we are able by attentive observation of an equation to discover a value of y , X suppose, which satisfies it, but which, as in the above example, does not turn out to be a singular solution.

In these cases, if we assume $y = X + \frac{1}{v}$, and substitute this quantity and its differential coefficients, we shall have an equation from which an expression for v involving an arbitrary constant may generally be determined.

As an example of the method of finding singular solutions of a differential equation without previous knowledge of the complete primitive; suppose we have

$$p^2 y + 2px - y = 0,$$

$$\text{where } p = \frac{dy}{dx}.$$

Then, differentiating so as to find $\frac{dp}{dx}$ and $\frac{dp}{dy}$,

$$\text{we have } 2p \frac{dp}{dx} y + 2p + 2x \frac{dp}{dy} = 0;$$

$$\therefore \frac{dp}{dx} = -\frac{p}{py+x};$$

$$\text{similarly, } \frac{dp}{dy} = \frac{1-p^2}{py+x};$$

and both these will equal infinity, if

$$py+x=0,$$

$$\text{that is, if } p = -\frac{x}{y};$$

and, substituting this value in the given differential equation, we obtain

$$x^2 + y^2 = 0;$$

which, as it satisfies the differential equation, and involves no constant, is a singular solution of it.

M. Poisson has shewn that a differential equation may be so prepared, as not to admit of singular solutions.

If $y = X$ be a singular solution of the equation $\frac{dy}{dx} = f(x, y)$, and m be the least index of z in the expansion of $\frac{d(X+z)}{dx}$, page 31; then, if the equation be divided by $(y-X)^m$, it will no longer admit of the singular solution in question.

For, putting $X+z$ for y in the value of $\frac{dy}{dx}$, and expanding as in page 31, we get

$$\frac{dX}{dx} + \frac{dz}{dx} = f(x, X) + Pz^m + Qz^n + \dots$$

where m is less than unity, $\therefore y = X$ is a singular solution.

Again, since $y = X$ satisfies the equation $\frac{dX}{dx} = f(x, X)$,

$$\therefore \frac{dz}{dx} = Pz^m + Qz^n + \dots$$

$$\text{or } \frac{dz}{dx} = P(y-X)^m + Q(y-X)^n + \dots$$

and this equation becomes identical, when $y = X$ or $z = 0$;

but, if we divide by z^m , we get

$$z^{-m} \frac{dz}{dx} = P + Q(y - X)^{n-m} \dots$$

which equation is no longer satisfied by the supposition $z = 0$; or, in other words, the differential equation divided by $(y - X)^m$ no longer admits of the singular solution $y = X$.

Ex. 1. The equation $\left(\frac{dy}{dx}\right)^2 - 4x\left(\frac{dy}{dx}\right) + 4y = 0$, has for its singular solution $y = x^2$, and the value of m in the required expansion is found to be $\frac{1}{2}$; therefore, if the equation be divided by $(x^2 - y)^{\frac{1}{2}}$, we shall find that it no longer admits of the singular solution.

For the equation may be put in the form

$$p^2 - 4xp + 4x^2 = 4x^2 - 4y;$$

$$\therefore p - 2x = 2\sqrt{x^2 - y};$$

therefore, dividing by the above, we have the equation

$$\frac{p - 2x}{\sqrt{x^2 - y}} = 2,$$

which is no longer satisfied by $y = x^2$, but gives the integral

$$2x + 2\sqrt{x^2 - y} = c.$$

Ex. 2. The equation $x dx + (y - \sqrt{x^2 + y^2 - a^2}) dy = 0$ is satisfied by the supposition $x^2 + y^2 - a^2 = 0$, which is equivalent to the two singular solutions

$$y = \sqrt{a^2 - x^2},$$

$$\text{and } y = -\sqrt{a^2 - x^2};$$

therefore, to deprive the equation of these, we must divide it by $(y - \sqrt{a^2 - x^2})^m \times (y + \sqrt{a^2 - x^2})^{m'}$.

But it is evident that the difference of sign of the radical cannot alter the value of m ; therefore $m' = m$, and the quantity by which our equation should be divided, is $(x^2 + y^2 - a^2)^m$.

To find the value of m , we have

$$\frac{dy}{dx} = \frac{x}{\sqrt{x^2 + y^2 - a^2 - y}},$$

therefore, putting $z + \sqrt{a^2 - x^2}$ for y , we get

$$p_1 = \frac{x}{\sqrt{2z\sqrt{a^2 - x^2} + z^2 - \sqrt{a^2 - x^2} - z}},$$

which, expanded according to powers of z , gives

$$p_1 = \frac{x}{\sqrt{a^2 - x^2}} + \frac{\sqrt{2z}}{(a^2 - x^2)^{\frac{3}{2}}} \dots$$

hence the value of m is $\frac{1}{2}$, and the equation is to be divided by $\sqrt{x^2 + y^2 - a^2}$, when it will become

$$\frac{xdx + ydy}{\sqrt{x^2 + y^2 - a^2}} = dy,$$

which is no longer satisfied by either of the singular solutions in question.

If $y = X$ be a singular solution of the equation $Mdx + Ndy = 0$, and if μ be the factor which makes this equation a complete differential; when X is put for y in the quantity μ , μ will become infinite.

For, suppose $V = 0$ to be the complete primitive; then, since $y = X$ is a singular solution, it will not satisfy the equation $V = 0$; therefore, it will not satisfy the equation

$$\frac{dV}{dx} dx + \frac{dV}{dy} dy = 0;$$

or, which is the same thing, it will not satisfy the equation

$$\mu \cdot (Mdx + Ndy) = 0. \dots (\alpha).$$

But, when X is put for y , it does make $Mdx + Ndy = 0$; therefore, it must make μ infinite, or else the equation (α) would be satisfied also.

COR. The above property of the singular solution has suggested a method of finding the integrating factor for a differential equation from the knowledge of its singular solutions.

We shall not occupy much time about this method, which has not proved very successful, but will refer the reader who is curious about it, to *Lacroix's* large work on the Differential and Integral Calculus, vol. ii, page 405.

The following examples will sufficiently shew the nature of the investigation.

Ex. 1. $x dx + (y - \sqrt{x^2 + y^2 - a^2}) dy = 0$

has a singular solution, which is $x^2 + y^2 - a^2 = 0$; therefore assume the integrating factor to be $(x^2 + y^2 - a^2)^n$, and multiplying our equation, would become

$$(x^2 + y^2 - a^2)^n \cdot x dx + \{y(x^2 + y^2 - a^2)^n - (x^2 + y^2 - a^2)^{n+\frac{1}{2}}\} dy = 0,$$

and applying the criterion of integrability, we find that we ought to have

$$2nxy(x^2 + y^2 - a^2)^{n-1} = 2nxy(x^2 + y^2 - a^2)^{n-1} - 2(n + \frac{1}{2})(x^2 + y^2 - a^2)^{n-\frac{1}{2}}$$

and this evidently will be the case if $n = -\frac{1}{2}$, but not otherwise.

Hence $\frac{1}{\sqrt{x^2 + y^2 - a^2}}$ is the integrating factor, and our equation is reduced to the form

$$\frac{x dx + y dy}{\sqrt{x^2 + y^2 - a^2}} = dy,$$

which is integrable, and gives

$$\sqrt{x^2 + y^2 - a^2} = y + c.$$

Ex. 2. $(a^2 - x^2) \frac{dy}{dx} + xy = a\sqrt{x^2 + y^2 - a^2}$ has a sin-

gular solution $x^2 + y^2 - a^2 = 0$, and is, moreover, satisfied by the supposition that $x^2 - a^2 = 0$;

therefore, assume the required factor to be

$$(x^2 - a^2)^m (x^2 + y^2 - a^2)^n,$$

and proceeding as in the last case, we shall find

$$\frac{1}{(x^2 - a^2) \sqrt{x^2 + y^2 - a^2}}$$

for the required factor; and, by means of it, our equation becomes

$$\frac{dy}{\sqrt{x^2 + y^2 - a^2}} + \left(\frac{xy}{(x^2 - a^2) \sqrt{x^2 + y^2 - a^2}} - \frac{a}{x^2 - a^2} \right) dx = 0,$$

which, it is easily seen, satisfies the criterion of integrability.

The curve represented by the singular solution, is the locus of the ultimate intersections of all the curves included in the complete integral, and touches them all.

The constant c being arbitrary, we may suppose the complete integral to represent an infinity of curves differing from each other only in the parameter c , which takes successive values indefinitely near each other.

Now the singular solution results from the elimination of c between the complete primitive

$$V = 0 = f(x, y, c),$$

$$\text{and } \frac{dv}{dc} = \phi(x, y, c) = 0;$$

also, the co-ordinates of intersection of any two of the curves corresponding to the values c and $(c + \delta c)$ of c , must satisfy simultaneously the equations

$$f(x, y, c) = V = 0,$$

$$\text{and } f\{x, y, (c + \delta c)\} = 0;$$

that is, they must satisfy the equations

$$V = 0,$$

$$\text{and } V + \frac{dv}{dc} \delta c + \frac{d^2v}{dc^2} \frac{(\delta c)^2}{1.2} + \dots$$

therefore, these co-ordinates must satisfy the equation

$$\frac{dV}{dc} + \frac{d^2V}{dc^2} \frac{\delta c}{1.2} + \dots = 0,$$

or, in the limit, since the values of c are consecutive, they must satisfy the equation

$$\frac{dV}{dc} = 0,$$

as well as $V = 0$.

And, therefore, if c be eliminated between these two, the result, which is the singular solution, and independent of c , must be satisfied by all the co-ordinates of all the intersections; and, therefore, it is the equation to the locus of all the intersections.

Again, for the same values of x and y , the values of $\frac{dy}{dx}$ obtained from the singular solution, and of $\frac{dy}{dx}$ obtained from the complete primitive, are equal; therefore, where the curves represented by the two equations meet, they have a common tangent; therefore the curve represented by the singular solution touches all the curves in the complete integral.

Hence, if we have given the equation to a curve $f(x, y, c) = 0$ to find the *bounding curve* to all the curves to which the equation $f(x, y, c) = 0$ may be made to belong, by varying c , we must eliminate c between the equations

$$f(x, y, c) = 0, \text{ and } \frac{d.f(x, y, c)}{dc} = 0.$$

CHAPTER IV.

ON EQUATIONS OF THE FIRST ORDER, BUT NOT OF THE FIRST DEGREE.

IN this chapter we shall first consider the equation of the n^{th} degree in its most general form, and then proceed to exemplify some particular methods applicable in particular cases.

SECT. 1. *On the Equation of the n^{th} degree.*

The most general form of this equation is

$$\left(\frac{dy}{dx}\right)^n + P \left(\frac{dy}{dx}\right)^{n-1} + Q \left(\frac{dy}{dx}\right)^{n-2} + \dots + T \frac{dy}{dx} + U = 0,$$

where $P, Q, \dots$ are functions of x and y . Solving this equation with respect to $\frac{dy}{dx}$, we should obtain, *generally*, n different values of that quantity, viz. :

$$\frac{dy}{dx} = p_1, \frac{dy}{dx} = p_2, \frac{dy}{dx} = p_3, \dots$$

and these n equations, being of the first order and first degree, are to be solved by the previous methods.

The integral of each of these will evidently satisfy the given equation, and will involve an arbitrary constant. The product of these integrals will also satisfy the equation, and, therefore, this product is the most general solution of the equation we are able to obtain.

It would appear, in the above process, that n arbitrary constants are introduced, and since the equation was only of the first order, this requires explanation. Now, since the constant, by the elimination of which the differential equation was obtained, must have entered the primitive in the n^{th} degree; it follows that it had n different values derivable from the primitive, and, therefore, the n constants introduced are only these n values of the constant eliminated.

If the quantities $P, Q, R, \dots$ be constant, the n values of $\frac{dy}{dx}$ must be all constant, therefore we may assume $\frac{dy}{dx} = m$, and substituting and solving the equation

$$m^n + Pm^{n-1} + \dots + Tm + U = 0 \dots (a),$$

we should obtain n integrals of the form

$$y = m_1x + c_1, \quad y = m_2x + c_2, \quad \dots$$

the product of which would be the general integral of the equation.

We might also have integrated at once the assumed equation $\frac{dy}{dx} = m$, getting $y = mx + c$, and therefore $m = \frac{y-c}{x}$, and substituting this value in (a) we have

$$\left(\frac{y-c}{x}\right)^n + P\left(\frac{y-c}{x}\right)^{n-1} + \dots + T\left(\frac{y-c}{x}\right) + U = 0.$$

It is evident in this solution of the equation that there would be n different values of c in terms of x and y , which corresponds with what has been said above, respecting the nature of the primitive giving rise to an equation of the n^{th} degree.

Ex. 1. Let the equation be

$$\left(\frac{dy}{dx}\right)^3 - 11\left(\frac{dy}{dx}\right)^2 + 26\left(\frac{dy}{dx}\right) - 24 = 0,$$

$$\therefore \text{assume } \frac{dy}{dx} = m,$$

and the equation gives

$$m^3 - 11m^2 + 26m - 24 = 0,$$

$$\text{but } \frac{dy}{dx} = m \text{ gives } y = mx + c,$$

$$\therefore m = \frac{y-c}{x};$$

therefore, the complete solution of the equation is

$$\left(\frac{y-c}{x}\right)^3 - 11\left(\frac{y-c}{x}\right)^2 + 26\left(\frac{y-c}{x}\right) - 24 = 0,$$

which is equivalent to

$$c^3 + c^2(3y - 11x) + c(22xy - 26x^2 - 3y^2) + y^3 - 11xy^2 + 26x^2y - 24x^2 = 0.$$

$$\text{Ex. 2. Let } \left(\frac{dy}{dx}\right)^2 - ax = 0,$$

therefore the factors of this equation are

$$\frac{dy}{dx} + a^{\frac{1}{2}} x^{\frac{1}{2}} = 0,$$

$$\text{and } \frac{dy}{dx} - a^{\frac{1}{2}} x^{\frac{1}{2}} = 0,$$

$$\therefore y - c = \pm \frac{2}{3} a^{\frac{1}{2}} x^{\frac{3}{2}},$$

therefore $(y - c)^2 = \frac{4}{9} ax^3$ is the complete integral of the equation.

$$\text{Ex. 3. } \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = a + b \frac{dy}{dx},$$

$$\text{then assume } \frac{dy}{dx} = m,$$

$$\therefore y = mx + c,$$

$$\therefore m = \frac{y-c}{x};$$

therefore, substituting for $\frac{dy}{dx}$, we have for the required integral

$$\sqrt{1 + \left(\frac{y-c}{x}\right)^2} = a + b \left(\frac{y-c}{x}\right).$$

There are, however, several forms of equations above the first degree which are more easily solved by particular methods.

SECT. 2. *On particular forms of Equations above the first degree.*

In the discussion of these, it is convenient to represent the differential coefficient $\frac{dy}{dx}$ by the letter p .

First, if an equation can be reduced to the form

$$y = px + P,$$

(called *Clairaut's form*), where P is a function of $\frac{dy}{dx}$;

then, differentiating,

$$dy = p dx + dp x + \frac{dP}{dp} dp,$$

$$\text{but } dy = p dx, \therefore 0 = dp \cdot \left(x + \frac{dP}{dp} \right);$$

therefore the differential equation is satisfied, either by supposing

first, $dp = 0$, which gives $p = c$,

and $\therefore y = cx + c'$,

which is the general integral of the proposed equation;

or, secondly, putting $x + \frac{dP}{dp} = 0$,

and eliminating p between this and the given equation, when we should obtain a relation between x and y , not involving an arbitrary constant, and being a singular solution.

Ex. Let $y = px + n\sqrt{1+p^2}$,

$$\therefore dy = p dx + dp \left(x + \frac{np}{\sqrt{1+p^2}} \right),$$

therefore the general solution of the equation is

$$y = ax + c,$$

while, to find the singular solution, we have

$$\begin{aligned}x + \frac{np}{\sqrt{1+p^2}} &= 0, \\ \therefore \frac{\sqrt{1+p^2}}{p} &= -\frac{n}{x}, \\ 1 + \frac{1}{p^2} &= \frac{n^2}{x^2}, \\ \therefore p^2 &= \frac{x^2}{n^2 - x^2};\end{aligned}$$

therefore, substituting the value of p in the proposed equation, we have for the singular solution

$$y^2 + x^2 = n^2.$$

Secondly, equations of the form

$$y = Px + P',$$

where P and P' are both functions of $\frac{dy}{dx}$, may be treated as follows :

$$\begin{aligned}dy &= Pdx + x \frac{dP}{dp} dp + \frac{dP'}{dp} dp; \\ \therefore (p - P) dx &= x \frac{dP}{dp} dp + \frac{dP'}{dp} dp;\end{aligned}$$

which is a linear equation between x and p , whence x is to be found in terms of p , and then p must be eliminated by means of the given equation.

Ex. Let $y = mpx + n\sqrt{1+p^3}$;

$$\therefore dy = mpx + mxdp + np^2dp(1+p^3)^{-\frac{1}{2}};$$

$$\therefore (m-1)pdx + mxdp + np^2dp(1+p^3)^{-\frac{1}{2}};$$

$$dx + \frac{m}{(m-1)p} xdp + \frac{n}{m-1} p(1+p^3)^{-\frac{1}{2}} dp = 0;$$

which is a linear equation between x and p , which becomes integrable if multiplied by the factor

$$e^{\int \frac{m}{m-1} \frac{dp}{p}} = p^{\frac{m}{m-1}},$$

$$\text{giving } p^{\frac{m}{m-1}}. x = -\frac{n}{m-1} \int p^{\frac{m-1}{m-1}} (1+p^3)^{-\frac{1}{2}} dp,$$

from which p is to be eliminated by means of the given equation.

Thirdly, if the equation be of the form

$$y = Px,$$

$$dy = Pdx + x \frac{dP}{dp} dp;$$

$$\therefore (p - P) dx = x \frac{dP}{dp} dp;$$

$$\therefore \frac{dx}{x} = \frac{\left(\frac{dP}{dp} dp\right)}{p - P};$$

whence x is to be found in terms of p , and then p must be eliminated.

$$\text{Ex.} \quad ydx - xdy = nx \sqrt{dx^2 + dy^2},$$

$$y = x. (p + n \sqrt{1+p^2}),$$

$$\therefore dy = pdx = dx (p + n \sqrt{1+p^2}) + x \left(dp + \frac{npdp}{\sqrt{1+p^2}} \right);$$

$$\therefore ndx \sqrt{1+p^2} + x \left(dp + \frac{npdp}{\sqrt{1+p^2}} \right) = 0;$$

$$\therefore \frac{dx}{x} + \frac{dp}{n \sqrt{1+p^2}} + \frac{pdp}{1+p^2} = 0;$$

$$\therefore \log_e x + \frac{1}{n} \log_e (p + \sqrt{1+p^2}) + \log_e \sqrt{1+p^2} + c = 0;$$

then, from the original equation, finding p in terms of x and y , and substituting in the last, we should have the required relation between x and y .

Fourthly, if our equation can be put in the form $x = P$,

where P is a function of $\frac{dy}{dx}$;

$$\text{we have } dx = \frac{dP}{dp} dp;$$

$$\therefore p dx = p \frac{dP}{dp} dp;$$

$$\therefore y = \int p \frac{dP}{dp} dp;$$

hence, having x and y in terms of p , a relation between x and y is to be found by eliminating p .

$$\text{Ex. } x dx + a dy = b \sqrt{dx^2 + dy^2},$$

$$\text{or } x = -ap + b \sqrt{1 + p^2};$$

$$\therefore dx = -a dp + \frac{b p dp}{\sqrt{1 + p^2}};$$

$$\therefore dy = -a p dp + \frac{b p^2 dp}{\sqrt{1 + p^2}};$$

whence y is to be found by integration, and p eliminated by means of the given equation.

An equation containing powers of $\frac{dy}{dx}$, but homogeneous with respect to x and y , may sometimes be solved conveniently as follows:

$$\text{Let } y dx - x dy = n x \sqrt{dx^2 + dy^2},$$

$$y - x \frac{dy}{dx} = n x \sqrt{1 + \frac{dy^2}{dx^2}},$$

$$y = x p + n x \sqrt{1 + p^2},$$

$$\frac{y}{x} = p + n \sqrt{1 + p^2},$$

$$\text{or } u = p + n \sqrt{1 + p^2} \dots (a).$$

$$\text{But } p dx = dy = d.xu = x du + u dx,$$

$$\therefore (p - u) dx = x du;$$

$$\therefore \frac{dx}{x} = \frac{du}{p - u};$$

$$\text{and from (a) } du = dp + \frac{n p dp}{\sqrt{1 + p^2}},$$

$$\text{and } p - u = -n \sqrt{1 + p^2};$$

$$\therefore \frac{dx}{x} = -\frac{pdp}{1+p^2} - \frac{dp}{n\sqrt{1+p^2}};$$

$$\therefore \log_e x = \frac{1}{2} \log_e \frac{1}{(1+p^2)} + \frac{1}{n} \log_e \frac{1}{(p + \sqrt{1+p^2})} + \log_e c;$$

$$\therefore x = \frac{c}{\sqrt{1+p^2}} (p + \sqrt{1+p^2})^{-\frac{1}{n}};$$

also, from (a) $y = x (p + n\sqrt{1+p^2})$,

$$\therefore y = c \frac{p + n\sqrt{1+p^2}}{\sqrt{1+p^2}} (p + \sqrt{1+p^2})^{-\frac{1}{n}}.$$

therefore, we have to eliminate p between these two equations.

Several of the above solutions are rather theoretical than practical in many apparently simple cases, in consequence of the difficulty or even impossibility of performing the elimination of p .

CHAPTER V.

ON EQUATIONS, IN WHICH THE DIFFERENTIAL COEFFICIENTS RISE ABOVE THE FIRST ORDER.

THE general integration of all equations, even of the first order, has not as yet been effected, but we are able, for the most part, to point out a *method* of solution applicable to any particular class, provided the requisite calculations be practicable. But, with the generality of equations of superior orders, we are unable to do this, the only class with which much progress has been made, being those denominated linear.

We shall, in this place, occupy ourselves with particular methods applicable to equations of particular forms, deferring the general discussion of linear equations to a subsequent section.

SECT. 1. *On the integration of some Equations of particular forms, of the second and higher orders.*

First, the equation $\frac{d^ny}{dx^n} = X$, where X is a function of x , may be solved by successive integration, and we have

$$\begin{aligned}\frac{d^{n-1}y}{dx^{n-1}} &= \int X dx + c = X_1 + c, \text{ suppose,} \\ \therefore \frac{d^{n-2}y}{dx^{n-2}} &= \int X_1 dx + \int c dx + c_1,\end{aligned}$$

$$= \int X_1 dx + cx + c_1;$$

and, proceeding in a similar manner, we should, after the n^{th} integration, have y in terms of x .

Thus, if we have

$$\begin{aligned}\frac{d^2y}{dx^2} &= -\frac{a^2}{(2ax - x^2)^{\frac{3}{2}}}; \\ \therefore \frac{dy}{dx} &= c - \int \frac{a^2 dx}{(2ax - x^2)^{\frac{3}{2}}}, \\ &= c + \frac{a - x}{\sqrt{2ax - x^2}}; \\ \therefore y &= cx + \int \frac{(a - x) dx}{\sqrt{2ax - x^2}}, \\ &= cx + \sqrt{2ax - x^2} + c_1,\end{aligned}$$

which is the equation to a curve of the second order.

Secondly, to integrate an equation of the form

$$\frac{d^n y}{dx^n} = F \frac{d^{n-1} y}{dx^{n-1}}.$$

Let the differential coefficient $\frac{d^{n-1} y}{dx^{n-1}} = p_{(n-1)}$,

$$\therefore \frac{dp_{(n-1)}}{dx} = F p_{(n-1)},$$

$$\therefore x = \int \frac{dp_{(n-1)}}{F p_{(n-1)}} + c \dots (1);$$

$$\text{and if } p_{(n-2)} = \frac{d^{n-2} y}{dx^{n-2}},$$

$$\begin{aligned}\text{we have } dp_{(n-2)} &= p_{(n-1)} dx, \\ &= \frac{p_{(n-1)} \cdot dp_{(n-1)}}{F p_{(n-1)}} \dots (2).\end{aligned}$$

Then we may eliminate $p_{(n-1)}$ between equation (1) and the integral of (2), thus finding

$$\frac{d^{n-2} y}{dx^{n-2}} = fx,$$

which is reduced to the last case.

Ex. 1. Let $a \frac{d^2y}{dx^2} \frac{d^3y}{dx^3} = \sqrt{1 + \left(\frac{d^2y}{dx^2}\right)^2}$,

therefore, putting $\frac{d^2y}{dx^2} = p_2$,

we have $\frac{dp_2}{dx} = \frac{\sqrt{1 + p_2^2}}{ap_2}$;

$$\therefore x = \int \frac{ap_2 dp_2}{\sqrt{1 + p_2^2}} = c + a \sqrt{1 + p_2^2} \dots (1),$$

$$\text{then } dp_1 = p_2 dx = \frac{ap_2^2 dp_2}{\sqrt{1 + p_2^2}},$$

$$\therefore p_1 = \int \frac{ap_2^2 dp_2}{\sqrt{1 + p_2^2}} + c \dots \dots \dots (2);$$

therefore, by elimination of p_2 between (1) and (2), we should have

$$\frac{dy}{dx} = \phi x,$$

whence y in terms of x .

We might also have taken

$$\begin{aligned} p_1 &= \int \frac{ap_2^2 dp_2}{\sqrt{1 + p_2^2}} + c, \\ &= \frac{ap_2^2}{2} \sqrt{1 + p_2^2} - \frac{a}{2} \log_e (p_2 + \sqrt{1 + p_2^2}) + c; \end{aligned}$$

$$\therefore dy = p_1 dx,$$

$$\begin{aligned} &= \left(\frac{ap_2^2}{2} \sqrt{1 + p_2^2} - \frac{a}{2} \log_e (p_2 + \sqrt{1 + p_2^2}) + c \right) \cdot \frac{ap_2 dp_2}{\sqrt{1 + p_2^2}}, \\ &= \left\{ \frac{a^2 p_2^2}{2} - \frac{a^2 p_2}{2\sqrt{1 + p_2^2}} \log_e (p_2 + \sqrt{1 + p_2^2}) + \frac{cap_2}{\sqrt{1 + p_2^2}} \right\} dp_2, \end{aligned}$$

whence, integrating, we should have

$$\begin{aligned} y &= \frac{a^2 p_2^3}{6} - \frac{a^2}{2} \sqrt{1 + p_2^2} \log_e (p_2 + \sqrt{1 + p_2^2}) \\ &\quad + \frac{a^2}{2} p_2 + ac \sqrt{1 + p_2^2} + c_1 \dots \dots \dots (3), \end{aligned}$$

and eliminating p_2 by means of equation (1), we should have y in terms of x .

In this example, p_2 may be eliminated without difficulty; because equation (1) gives us

$$\sqrt{1 + p_2^2} = \frac{x - c}{a},$$

$$\therefore 1 + p_2^2 = \frac{(x - c)^2}{a^2}, \text{ and } p_2 = \frac{\sqrt{(x - c)^2 - a^2}}{a}.$$

Ex. 2. The equation to the catenary, when the string is extensible, is (*Whewell's Mech.* art. 140)

$$\frac{dp}{dx} = \frac{\sqrt{1 + p^2}}{c + c^2 \epsilon \sqrt{1 + p^2}};$$

$$\therefore dx = \frac{cdp}{\sqrt{1 + p^2}} + c^2 \epsilon dp,$$

$$x = c \log_e (p + \sqrt{1 + p^2}) + c^2 \epsilon p \dots (1),$$

$$y = c \sqrt{1 + p^2} + \frac{1}{2} c^2 \epsilon p^2 \dots (2),$$

the integrals being taken, so that when $p = 0$, $x = 0$, and $y = c$.

The second method of solution has here been adopted; because, from equation (1) we could not obtain p in terms of x , while from equation (2) we can find p in terms of y , and substitute the value in equation (1), thus obtaining x in terms of y .

Ex. 3. To find the curve in which the radius of curvature is constant.

This condition gives us the differential equation

$$\frac{(1 + p^2)^{\frac{3}{2}}}{\left(\frac{dp}{dx}\right)} = a;$$

$$\therefore dx = \frac{adp}{(1 + p^2)^{\frac{3}{2}}},$$

$$\text{whence } x = \frac{ap}{\sqrt{1 + p^2}} + c,$$

$$\text{and } dy = p dx = \frac{ap dp}{(1+p^2)^{\frac{3}{2}}};$$

$$\therefore y = -\frac{a}{\sqrt{1+p^2}} + c_1;$$

therefore, eliminating p , we have

$$\begin{aligned}(x-c)^2 + (y-c_1)^2 &= \frac{a^2 p^2}{1+p^2} + \frac{a^2}{1+p^2}, \\ &= a^2 \cdot \frac{1+p^2}{1+p^2}, \\ &= a^2,\end{aligned}$$

which is the equation to a circle, the only curve which satisfies the condition required.

Thirdly, another form which is frequently integrable, is

$$\frac{d^n y}{dx^n} = F \frac{d^{n-2} y}{dx^{n-2}},$$

$$\text{or } \frac{d^2 p_{(n-2)}}{dx^2} = F p_{(n-2)};$$

$$\therefore 2dp_{(n-2)} \frac{d^2 p_{(n-2)}}{dx^2} = 2F p_{(n-2)} dp_{(n-2)},$$

$$\therefore d. \left(\frac{dp_{(n-2)}}{dx} \right)^2 = 2F p_{(n-2)} dp_{(n-2)},$$

$$\begin{aligned}\therefore \frac{dp_{(n-2)}}{dx} &= \sqrt{\int 2F p_{(n-2)} dp_{(n-2)}} \\ &= \psi p_{(n-2)} + c;\end{aligned}$$

$$\therefore dx = \frac{dp_{n-2}}{\psi p_{(n-2)} + c} \dots (1),$$

$$\therefore x = \int p_{(n-2)},$$

$$\therefore \frac{d^{n-2} y}{dx^{n-2}} = X,$$

which belongs to the first case.

Ex. 1. Let $\frac{d^2 y}{dx^2} = \frac{1}{\sqrt{ay}},$

$$\therefore 2dy \frac{d^2 y}{dx^2} = \frac{2dy}{\sqrt{ay}},$$

$$\begin{aligned}
 \therefore \left(\frac{dy}{dx}\right)^2 &= \frac{4}{a} \sqrt{ay} + c_1, \\
 &= \frac{4}{\sqrt{a}} (\sqrt{y} + c) \text{ suppose,} \\
 \frac{dy}{dx} &= \frac{2}{a^{\frac{1}{2}}} \sqrt{c + \sqrt{y}}, \\
 \therefore dx &= \frac{a^{\frac{1}{2}}}{2} \frac{dy}{\sqrt{c + \sqrt{y}}},
 \end{aligned}$$

an equation of the first order, in which the variables are separated, and may be integrated by putting

$$c + \sqrt{y} = z.$$

Ex. 2.

$$\begin{aligned}
 \frac{d^4y}{dx^4} &= \frac{d^2y}{dx^2}, \\
 \text{put } \frac{d^2y}{dx^2} &= q, \therefore \frac{d^2q}{dx^2} = q; \\
 \therefore 2dq \frac{d^2q}{dx^2} &= 2q dq; \\
 \therefore \left(\frac{dq}{dx}\right)^2 &= q^2 + a; \\
 \therefore \frac{dq}{dx} &= \sqrt{q^2 + a};
 \end{aligned}$$

whence $x = \log_e (q + \sqrt{q^2 + a}) + \log_e b$,

$$\therefore \text{ we get } \frac{e^x}{b} = q + \sqrt{q^2 + a};$$

$$\therefore q = \frac{e^x}{2b} - \frac{ab}{2} e^{-x},$$

$$\text{or } \frac{dp}{dx} = \frac{e^x}{2b} - \frac{ab}{2} e^{-x};$$

$$\therefore \text{ integrating } p = \frac{e^x}{2b} + \frac{ab}{2} e^{-x} + c,$$

$$\therefore y = \frac{e^x}{2b} - \frac{ab}{2} e^{-x} + cx + d,$$

when a, b, c, d , are the four arbitrary constants.

Fourthly, suppose our equation be of the form

$$f\left(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0,$$

or, as we shall in future represent the differential coefficients

$$f(x, p, q) = 0.$$

Then, if we can integrate the equation $f\left(x, p, \frac{dp}{dx}\right) = 0$, we shall either find p in terms of x , or x in terms of p . Suppose, first, that we have found

$$p = X + c, \text{ where } X \text{ is a function of } x;$$

$$\therefore dy = Xdx + cdx,$$

$$\therefore y = \int Xdx + cx + c_1$$

is the solution of the given equation; but if, on the other hand, we have found

$$x = \phi p = (P + c);$$

$$\text{we have also } dy = p dx,$$

$$\therefore y = px - \int x dp$$

$$= px - \int (P + c) dp.$$

Hence x and y are both found in terms of p , and, therefore, p must be eliminated.

Ex. 1. Let the equation be

$$\frac{dy}{dx} + x \frac{d^2y}{dx^2} = \frac{a^2 + 2x^2}{\sqrt{a^2 + x^2}},$$

$$\text{or } p dx + x dp = \frac{a^2 + 2x^2}{\sqrt{a^2 + x^2}} dx;$$

$$\therefore px = \int \sqrt{a^2 + x^2} dx + \int \frac{x^2 dx}{\sqrt{a^2 + x^2}},$$

$$= x \sqrt{a^2 + x^2} + c;$$

$$\therefore p = \sqrt{a^2 + x^2} + \frac{c}{x};$$

$$\therefore dy = dx \sqrt{a^2 + x^2} + \frac{cdx}{x};$$

$$\therefore y = \frac{x}{2} \sqrt{a^2 + x^2} + \frac{a^2}{2} \log_e (x + \sqrt{a^2 + x^2}) + c \log_e x + c_1,$$

$$\text{or} = \frac{x}{2} \sqrt{a^2 + x^2} + \log_e (x + \sqrt{a^2 + x^2})^{\frac{a^2}{2}} + \log_e x^c + c_1.$$

$$\text{Ex. 2. } \frac{a^2}{2x} = \frac{(1+p^2)^{\frac{3}{2}}}{\left(\frac{dp}{dx}\right)},$$

$$\therefore 2x(1+p^2)^{\frac{3}{2}} dx = a^2 dp,$$

$$\therefore 2xdx = \frac{a^2 dp}{(1+p^2)^{\frac{3}{2}}};$$

$$\therefore \text{integrating } x^2 + c = \frac{a^2 p}{\sqrt{1+p^2}},$$

$$\therefore \left(\frac{x^2 + c}{a^2}\right)^2 = \frac{p^2}{1+p^2},$$

$$\therefore \frac{a^4}{(x^2 + c)^2} = \frac{1}{p^2} + 1,$$

$$\therefore \frac{1}{p^2} = \frac{a^4 - (x^2 + c)^2}{(x^2 + c)^2},$$

$$\therefore p = \frac{x + c}{\sqrt{a^4 - (x^2 + c)^2}},$$

$$\therefore y = \int \frac{(x^2 + c) dx}{\sqrt{a^4 - (x^2 + c)^2}},$$

whence y may be found in term of x .

As an example of the second method, let

$$3(1-p^2)^{-4} p \frac{dp}{dx} + x = 0,$$

which is an equation, in which the variables are at once separated, and has for its integral

$$x = \frac{1}{(1-p^2)^{\frac{3}{2}}} + c.$$

Then, since $dy = p dx$,

$$y = px - \int x dp;$$

$$\begin{aligned}\therefore y &= px - \int \frac{dp}{(1-p^2)^{\frac{3}{2}}} - \int c dp, \\ &= px - \frac{p}{\sqrt{1-p^2}} - cp + c_1;\end{aligned}$$

and from the first equation

$$(1-p^2) = (x-c)^{-\frac{2}{3}},$$

whence the value of p in terms of x may be substituted in the value of y .

Fifthly, suppose our equation of the form

$$f\left(y, p, \frac{dp}{dx}\right) = 0,$$

we shall, if the equation is integrable, find

$$p = Y, \text{ or } y = P.$$

Suppose, first, that we have $p = Y$,

$$\therefore \frac{dy}{dx} = Y,$$

$$\therefore dx = \frac{dy}{Y};$$

which, being integrated, we have x in terms of y .

Secondly, if we get $y = P$,

then, because $p dx = dy$,

$$\therefore dx = \frac{dy}{p};$$

$$\begin{aligned}\therefore x &= \frac{y}{p} + \int \frac{y dp}{p^2}, \\ &= \frac{y}{p} + \int \frac{P dp}{p^2}, \\ &= \frac{P}{p} + \int \frac{P dp}{p^2};\end{aligned}$$

thus x and y are both found in terms of p , which, therefore, is to be eliminated.

Ex. Let $\frac{dp}{dx}(yp + a) = p(1 + p^2),$

$$\therefore dp(yp + a) = dy(1 + p^2);$$

$$\therefore dy - \frac{p}{1+p^2} y dp = \frac{adp}{1+p^2},$$

which is a linear equation, which becomes integrable by multiplying by the factor $\frac{1}{\sqrt{1+p^2}}$, and gives for integral

$$\frac{y}{\sqrt{1+p^2}} = \frac{ap}{\sqrt{1+p^2}} + c,$$

$$\therefore y = ap + c\sqrt{1+p^2} \dots (1);$$

$$\text{then } dx = \frac{dy}{p},$$

$$\therefore x = \frac{y}{p} + \int \frac{y dp}{p^2},$$

$$= \frac{ap + c\sqrt{1+p^2}}{p} + \int \frac{(ap + c\sqrt{1+p^2}) dp}{p^2};$$

then, taking the integral, we should eliminate p by substituting its value from equation (1).

Sometimes we meet with equations in which the differential coefficients have been taken, with respect to the arc of the curve, as the independent variable; in this case, the usual transformation must be applied so as to have x for the independent variable, before we proceed with the integration.

$$\text{Ex. 1. } \left(\frac{dy}{ds}\right)^2 = a \frac{dx}{ds} \cdot \frac{d^2x}{ds^2},$$

we have, by the rules of Differential Calculus,

$$\frac{dy}{ds} = \frac{\left(\frac{dy}{dx}\right)}{\left(\frac{ds}{dx}\right)} = \frac{p}{\sqrt{1+p^2}},$$

$$\frac{dx}{ds} = \frac{1}{\left(\frac{ds}{dx}\right)} = \frac{1}{\sqrt{1+p^2}},$$

$$\frac{d^2x}{ds^2} = \frac{dx}{ds} \cdot \frac{d\left(\frac{dx}{ds}\right)}{dx},$$

$$\begin{aligned}
 &= \frac{1}{\sqrt{1+p^2}} \cdot \frac{d}{dx} (1+p^2)^{-\frac{1}{2}}, \\
 &= -\frac{dp}{dx} \frac{p}{(1+p^2)^{\frac{3}{2}}};
 \end{aligned}$$

therefore, substituting these values, our equation becomes

$$\begin{aligned}
 \frac{p^2}{1+p^2} + \frac{ap}{(1+p^2)^{\frac{3}{2}}} \frac{dp}{dx} &= 0, \\
 \therefore \frac{dx}{dp} &= -\frac{a}{p(1+p^2)^{\frac{3}{2}}},
 \end{aligned}$$

$$\text{whence } x = c - \frac{a}{\sqrt{1+p^2}} + a \log \frac{(1+\sqrt{1+p^2})}{p},$$

$$\begin{aligned}
 \text{and } dy &= p dx, \\
 &= -\frac{adp}{(1+p^2)^{\frac{3}{2}}},
 \end{aligned}$$

$$\therefore y = c - \frac{ap}{\sqrt{1+p^2}},$$

whence p may be found in terms of y , and substituted in the equation for x .

$$\text{Ex. 2. } \left(\frac{dy}{ds}\right) = \frac{d^2x}{ds^2} a \tan^{-1} \frac{\left(\frac{dy}{ds}\right)}{\left(\frac{dx}{ds}\right)},$$

taking the values of $\frac{dy}{ds}$ and $\frac{d^2x}{ds^2}$, as found in the last example, and observing that

$$\frac{\left(\frac{dy}{ds}\right)}{\left(\frac{dx}{ds}\right)} = \frac{dy}{dx} = p,$$

our equation becomes

$$\begin{aligned}
 \frac{p}{\sqrt{1+p^2}} &= -\frac{dp}{dx} \frac{ap}{(1+p^2)^2} \tan^{-1} p, \\
 \therefore dx &= -\frac{adp}{(1+p^2)^{\frac{3}{2}}} \tan^{-1} p,
 \end{aligned}$$

$$dy = - \frac{apdp}{(1+p^2)^{\frac{3}{2}}} \tan^{-1} p ;$$

the first gives us

$$x = c - \frac{ap}{\sqrt{1+p^2}} \tan^{-1} p - \frac{ap}{\sqrt{1+p^2}},$$

and the second

$$y = c_1 + \frac{a}{\sqrt{1+p^2}} \tan^{-1} p - \frac{a}{\sqrt{1+p^2}}.$$

Then p may be eliminated from these equations, for multiplying the second by p , and adding, we have

$$x + py = c + c_1 p - \frac{2ap}{\sqrt{1+p^2}},$$

from which equation p being determined, may be substituted in either of the above.

SECT. 2. *General discussion of Linear Equations.*

It has already been stated, that when an equation contains only the first power of y , and its differential coefficients, it is called *linear*; and the linear equation of the first order has been integrated.

We have seen, however, that in obtaining the complete integral of an equation above the first order, the constants are not merely added, but that they are generally involved with the variables.

Hence, although we may have found a function of x and y , which satisfies an equation of the n^{th} order, it can by no means be considered the complete solution, unless it contains n arbitrary constants properly involved.

It has been proved in Chapter I. that every equation of the n^{th} order, has n particular integrals, each involving an arbitrary constant; but we have yet to shew in what way the complete primitive, containing n different arbitrary constants, and satisfying the differential equation, is to be found from the particular integrals.

This we shall proceed to do for linear equations of the n^{th} order.

If $p, q, r \dots$ be n functions of x , which substituted for y satisfy the equation

$$\frac{d^n y}{dx^n} + A \frac{d^{n-1} y}{dx^{n-1}} + \dots + K \frac{dy}{dx} + Ly = 0 \dots (a).$$

the coefficients being constant or functions of x ; then the complete integral will be $y = ap + bq + cr + \dots$ where $a, b, c, \dots$ are n arbitrary constants.

(We shall here represent the differential coefficients taken with respect to x by accents, and the n^{th} differential coefficient of $y \dots$ by $y^{(n)}$).

Then, supposing $y = ap + bq + cr + \dots$

we have $y' = ap' + bq' + cr' + \dots$

$y'' = ap'' + bq'' + cr'' + \dots$

$\dots = \dots$

$y^{(n-1)} = ap^{(n-1)} + bq^{(n-1)} + cr^{(n-1)} + \dots$

$y^{(n)} = ap^{(n)} + bq^{(n)} + cr^{(n)} + \dots$

and substituting these values of the differential coefficients in the proposed equation, the result ought to $= 0$; but the result is

$$\begin{aligned} & a(p^{(n)} + Ap^{(n-1)} + \dots + Kp' + Lp) \\ & + b(q^{(n)} + Aq^{(n-1)} + \dots + Kq' + Lq) \\ & + c(r^{(n)} + Ar^{(n-1)} + \dots + Kr' + Lr), \\ & + \dots \end{aligned}$$

and on the hypothesis that $p, q, r, \dots$ are particular integrals, this would evidently $= 0$, because each term merely consists of a constant multiplied into the expression (a) with one of the particular integrals in the place of y ; therefore each term is separately $= 0$.

Hence the assumption that we have made satisfies the differential equation, and n arbitrary constants are introduced; therefore $y = ap + bq + cr + \dots$ is the complete integral.

To deduce the complete primitive of the equation

$$y^{(n)} + Ay^{(n-1)} + By^{(n-2)} + \dots + Ky' + Ly = X$$

from the particular integrals of the equation

$$y^{(n)} + Ay^{(n-1)} + \dots + Ky' + Ly = 0,$$

the coefficients $A, B, C, \dots X$ being either constant or functions of x .

Suppose $p, q, r, \dots$ the n particular integrals of the latter equation, and assume the complete integral of the equation to be

$$y = ap + bq + cr + \dots$$

where $a, b, c, \dots$ are no longer arbitrary constants, but functions of x , which it is our object to determine.

Then, differentiating,

$$y' = ap' + bq' + cr' + \dots + a'p + b'q + c'r + \dots$$

$$\text{then assume } a'p + b'q + c'r + \dots = 0;$$

also

$$y'' = ap'' + bq'' + \dots + a''p + b''q + \dots + a'p' + b'q' + c'r' + \dots$$

$$\text{then assume } a'p' + b'q' + c'r' + \dots = 0;$$

and, similarly, proceeding to $(n-1)$ differentiations, and assuming, in each, the sum of the terms involving $a', b', c', \dots$ to $= 0$; and in the value of $y^{(n)}$ assuming them $= X$, we have n equations to determine the n quantities $a', b', c', \dots$

viz:

$$pa' + qb' + rc' \dots = 0, \quad p'a' + q'b' + r'c' \dots = 0,$$

$$\dots = 0; \quad p^{(n-1)}a' + q^{(n-1)}b' + r^{(n-1)}c' = X;$$

and having assumed these equations, it is evident that all the terms involving $a'', b'', c'', \dots a''', b''', c''' \dots$ must $= 0$; and, therefore, the differential coefficients are reduced to

$$y = ap + bq + cr + \dots$$

$$y' = ap' + bq' + cr' + \dots$$

$$= \dots$$

$$y^{(n)} = ap^{(n)} + bq^{(n)} + cr^{(n)} + \dots$$

And substituting in the given equation, it becomes

$$\left. \begin{aligned} &+ a(p^{(n)} + Ap^{(n-1)} + \dots + Kp' + Lp) \\ &+ b(q^{(n)} + Aq^{(n-1)} + \dots + Kq' + Lq) \\ &+ \text{terms assumed} = 0 \\ &+ \text{terms assumed} = X \end{aligned} \right\} = X;$$

and this will evidently be true, because each term, except the last in the above equation, has for a factor the quantity $(y^{(n)} + Ay^{(n-1)} + \dots + Ky' + Ly)$, with one of the particular integrals in the place of y ; therefore, when the functions $a, b, c, \dots$ are determined according to our assumptions, the whole is equivalent to $X = X$.

Hence our assumptions satisfy the given equation; also, from the n assumed equations, we may find the n quantities $\frac{da}{dx}, \frac{db}{dx}, \dots$ whence, $a, b, c, \dots$ being found by integration, n arbitrary constants will be introduced; therefore, our assumption gives us the complete primitive.

This method of solution is called the *variation of parameters*.

Conversely, if by any means we can obtain the complete integral of the equation

$$\frac{d^n y}{dx^n} + P \frac{d^{n-1} y}{dx^{n-1}} + \dots + K \frac{dy}{dx} + Ly = 0;$$

then the *part* connected with each of the n arbitrary constants will of itself satisfy the differential equation, and, therefore, will be one of its particular integrals; for since all the constants are arbitrary, *any* $(n - 1)$ of *these parts* may be supposed to evanesce, and the remaining *part* of the integral must still satisfy the differential equation; that is, each part separately of the integral must satisfy the equation, or, in other words, be a particular integral of it.

This observation will be found important when we come to the application of the method to the solution of linear equations with *variable* coefficients.

The particular integrals from which the general solution is deduced, can only be found for a very limited number of equations, among which are those with *constant coefficients*.

To find the complete solution of the general linear equation

$$y^{(n)} + Ay^{(n-1)} + \dots + Ky' + Ly = 0,$$

the coefficients being constant.

Assume $y = e^{hx}$, where h is a quantity (not a function of x) to be determined,

$$\therefore y' = he^{hx}, y'' = h^2e^{hx},$$

$$\dots\dots\dots y^{(n)} = h^ne^{hx};$$

therefore, substituting these values in the given equation, and dividing the whole by the common factor e^{hx} , we have, for the determination of h , the equation

$$h^n + Ah^{n-1} + \dots + Kh + L = 0 \dots (\beta).$$

1st. Suppose the roots of the equation to be all *possible and unequal*, and let them be $\alpha, \beta, \gamma, \dots$; then the n particular integrals are $e^{\alpha x}, e^{\beta x}, e^{\gamma x}, \dots$ therefore, the complete solution is

$$y = ae^{\alpha x} + be^{\beta x} + ce^{\gamma x} + \dots$$

2nd. Suppose the equation for h to contain *impossible roots*, then these must be in pairs, and of the form

$$\alpha' + \beta' \sqrt{-1}, \text{ and } \alpha' - \beta' \sqrt{-1},$$

which gives the particular integrals

$$\begin{aligned} & ae^{(\alpha' + \beta' \sqrt{-1})x} + be^{(\alpha' - \beta' \sqrt{-1})x} \\ &= e^{\alpha' x} \cdot \{ ae^{\beta' x \sqrt{-1}} + be^{-\beta' x \sqrt{-1}} \}, \\ &= e^{\alpha' x} \cdot \left\{ (a+b) \cdot \frac{e^{\beta' x \sqrt{-1}} + e^{-\beta' x \sqrt{-1}}}{2} + (a-b) \sqrt{-1} \left(\frac{e^{\beta' x \sqrt{-1}} - e^{-\beta' x \sqrt{-1}}}{2\sqrt{-1}} \right) \right\} \\ &= e^{\alpha' x} \cdot \{ C_1 \cos \beta' x + C_2 \sin \beta' x \}; \end{aligned}$$

representing the arbitrary constants

$$(a+b), (a-b) \sqrt{-1}, \text{ by } C_1, C_2, \text{ respectively;}$$

and similarly for any other pairs of impossible roots the corresponding part of the complete integral may be found.

3rd. Suppose the equation for h to contain (m) equal roots each $= a$, it is evident that the part of the integral

$$y = ae^{ax} + be^{\beta x} + ce^{\gamma x} + \dots$$

would be reduced to $y = (a + b + c + \dots) e^{ax}$, and $a + b + c \dots$ can only be considered as one arbitrary constant; and, therefore, this would only give us one particular integral; and, therefore, on the whole, we should only have $(n - m + 1)$ particular integrals.

To avoid this failure, assume $y = e^{hx}u$;

$$\therefore \frac{dy}{dx} = e^{hx} \left\{ hu + \frac{du}{dx} \right\} = e^{hx} (hu + u'),$$

$$\frac{d^2y}{dx^2} = e^{hx} \{ h^2u + 2hu' + u'' \}.$$

$$\dots = \dots$$

$$\frac{d^ny}{dx^n} = e^{hx} \{ h^nu + nh^{n-1}u' + \dots + u^{(n)} \}.$$

Then, substituting these values in the given equation, it may be put in the form

$$\begin{aligned} & u \cdot (h^n + Ah^{n-1} + Bh^{n-2} + \dots + L) \\ & + u' \{ nh^{n-1} + (n-1)Ah^{n-2} + \dots + K \} \\ & + u'' \{ n(n-1)h^{n-2} + \dots + \} + \dots \end{aligned}$$

omitting the common factor e^{hx} ; or, in other symbols, (representing the quantity $h^n + Ah^{n-1} + \dots$ by V , and its differential coefficients taken with respect to h by V' , V'' , ...) the given equation takes the form

$$uV + u'V' + \dots + u^{(m)}V^{(m)} + \dots + u^{(n)}V^{(n)} = 0 \dots (A):$$

Now, since the equation $V = 0$ contains m equal values of h , therefore $(m - 1)$ of its differential coefficients, taken with respect to h , must of themselves equal nothing; therefore the m first terms in equation (A) are equal to nothing; and the others will equal nothing if $u^{(m)}$, $u^{(m+1)}$, $\dots$ each $= 0$; that is, if u is a function of x of $(m - 1)$ dimensions, that is,

if $u = (a + bx + cx^2 + \dots + gx^{m-1})$,

$a, b, c, \dots$ being constant;

and the supposition that u is of this form, introduces the requisite number of arbitrary constants, and therefore

$$e^{ax} \{a + bx + \dots + gx^{m-1}\}$$

is the part of the integral corresponding to the m equal roots, the other part being found as in the first case.

Hence the complete solution is

$$y = e^{ax} \{a + bx + cx^2 + \dots + gx^{m-1}\} + b_1 e^{\beta x} + c_1 e^{\gamma x} + \dots$$

$\beta, \gamma, \dots$ being the unequal roots.

The above method for the equal roots has been given, because it appears the most perfect; there is, however, another which is much simpler, but probably less satisfactory.

If two roots of the equation for h become equal, then the corresponding part of the integral is $Ce^{ax} + C'e^{ax}$; but suppose these roots, instead of being absolutely equal, to differ by the small indeterminate quantity κ , therefore the part of the integral

$$= e^{ax} \{C + C'e^{\kappa x}\},$$

$$= e^{ax} \left\{ C + C' \left(1 + \kappa x + \frac{\kappa^2 x^2}{1.2} + \dots \right) \right\},$$

$$= e^{ax} \left\{ C + C' + C'\kappa x + \frac{C'\kappa^2 x^2}{1.2} + \dots \right\},$$

then let $C + C' = C_1$, $C'\kappa = C_2$; therefore this part becomes

$$e^{ax} \left\{ C_1 + C_2 x + \frac{C_2 \kappa^2 x^2}{\kappa 1.2} + \dots \right\};$$

$$\text{or if } \kappa = 0, \text{ it becomes } e^{ax} \{C_1 + C_2 x\},$$

which is the same form as that previously deduced, and, proceeding on similar principles, we should find the integral, when three or more roots were equal.

The circumstance of $C'\kappa$ being put $= C_2$ when $\kappa = 0$, may appear unsatisfactory; but C' being by hypothesis arbitrary, we are at liberty to consider it infinite, in which case $C'\kappa$ may be finite.

4th. If there are *equal impossible roots* in the equation, they must occur in equal pairs; that is, we must have two roots, each $= a' + \beta' \sqrt{-1}$, and two, each $= a' - \beta' \sqrt{-1}$; therefore the part of the integral corresponding to these four roots will be

$$e^{ax} \cdot \{ (a + bx) e^{\beta' x \sqrt{-1}} + (c + dx) e^{-\beta' x \sqrt{-1}} \},$$

which, by the same transformation as that formerly used for impossible roots, gives us

$$e^{ax} \cdot \{ (C_1 + C_2 x) \cos \beta' x + (C_3 + C_4 x) \sin \beta' x \};$$

and, similarly, if there were three equal pairs of impossible roots, the part of the integral would be

$$e^{ax} \cdot \{ (C_1 + C_2 x + C_3 x^2) \cos \beta' x + (C_4 + C_5 x + C_6 x^2) \sin \beta' x \},$$

and so on, for any number of equal pairs of impossible roots.

When the solution of the equation

$$y^{(n)} + Ay^{(n-1)} + \dots + Ky' + Ly = 0 \dots (a)$$

with constant coefficients has been obtained, that of the equation

$$y^{(n)} + Ay^{(n-1)} + \dots + Ky' + Ly = X \dots (\beta)$$

may be deduced from it by the method in page 62; and hence, if an equation of the form (β) were given, it would be first necessary to solve an equation of the form (a) , and from the solution to deduce that of (β) .

The solution of linear equations above the second order, in which the coefficients are functions of x , has scarcely been attempted, and even with the second order it is not always practicable.

It has been shewn that the solution of the equation

$$\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = X \dots (1)$$

can be deduced from the particular integrals of the equation

$$\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = 0 \dots (2).$$

To solve the latter, in the cases in which it is possible,

$$\text{assume } y = e^{\int u dx}, \therefore y' = ue^{\int u dx};$$

$$\text{and } y' = u^2 e^{\int u dx} + \frac{du}{dx} e^{\int u dx};$$

then, substituting these values in (2), and dividing by the common factor $e^{\int u dx}$, we have the equation

$$u^2 + \frac{du}{dx} + Pu + Q = 0 \dots (3), \quad \checkmark$$

which being of the first order may sometimes be integrated, when we shall find a value of u in terms of x containing one constant; and, in the integration for obtaining $\int u dx$, a second constant will be introduced; therefore $y = e^{\int u dx}$, which satisfies the differential equation (2), contains two constants, and therefore is the complete solution of it.

. *The cases in which the reducing equation (3) can be solved, are*

1st. If $Q = 0$, when it takes the form

$$u^2 dx + P u dx + du = 0;$$

$$\text{therefore, putting } u = \frac{1}{v}, du = -\frac{dv}{v^2},$$

and substituting, we have the linear equation

$$dv - P v dx = dx;$$

$$\therefore e^{-\int P dx} dv - e^{-\int P dx} P v dx = e^{-\int P dx} dx;$$

$$\therefore v e^{-\int P dx} = C + \int \frac{dx}{e^{\int P dx}};$$

$$\therefore v = e^{\int P dx} \cdot \left\{ C + \int \frac{dx}{e^{\int P dx}} \right\};$$

$$\therefore u = \frac{1}{v} \text{ is known.}$$

2nd. If $P = 2\sqrt{Q}$, the equation takes the form

$$(u + \sqrt{Q})^2 dx + du = 0;$$

$$\text{then, putting } u + \sqrt{Q} = z,$$

$$du + \frac{d\sqrt{Q}}{dx} dx = dz;$$

therefore, substituting, the equation becomes

$$z^2 dx + dz - \frac{d\sqrt{Q}}{dx} dx = 0;$$

and this equation is always integrable if $\frac{d\sqrt{Q}}{dx}$ is constant, which gives us the condition that $\sqrt{Q} = a + bx$.

Therefore the equation may be integrated if $P = 2(a + bx)$, and $Q = (a + bx)^2$.

3rd. If $P = \frac{A}{(a + bx)}$, and $Q = \frac{B}{(a + bx)^2}$, the reducing equation becomes

$$\frac{du}{dx} + u^2 + \frac{A}{(a + bx)} u + \frac{B}{(a + bx)^2} = 0.$$

Then assume $u = \frac{m}{(a + bx)}$, where m is indeterminate;

$$\therefore \frac{du}{dx} = -\frac{mb}{(a + bx)^2};$$

therefore, substituting and omitting the common factor $\frac{1}{(a + bx)^2}$,

we have, for the determination of m , the equation

$$m^2 + m(A - b) + B = 0;$$

whence two values of m being found, the two values of u and therefore of $e^{\int u dx}$ will be determined, and these will be the two particular integrals.

4th. The equation (3) may be solved if the coefficients are such that $Q = \frac{dP}{dx}$, for it then becomes

$$u^2 + Pu + \frac{dP}{dx} + \frac{du}{dx} = 0,$$

which equation is evidently satisfied by the supposition $u = -P$; but this integral will not satisfy it if a constant is added, and therefore it is either a singular solution or a particular integral corresponding to an infinite value of the arbitrary constant. Applying the test as in page 33, example 2, we find that it is a particular integral.

To find the general integral, assume

$$u = \frac{1}{v} - P;$$

therefore, substituting, we have

$$\begin{aligned}\left(\frac{1}{v} - P\right)^2 + \frac{P}{v} - P^2 + \frac{dP}{dx} - \frac{dv}{v^2 dx} - \frac{dP}{dx} &= 0, \\ \text{or } \frac{1}{v^2} - \frac{P}{v} - \frac{dv}{v^2 dx} &= 0; \\ \therefore 1 - Pv - \frac{dv}{dx} &= 0;\end{aligned}$$

which is a linear equation of the first order, from which v may be found, and an arbitrary constant will be introduced;

$$\therefore u = \frac{1}{v} - P, \text{ is known,}$$

and will be the complete solution of the reducing equation in this case.

We will now give some instances of the application of the methods above investigated.

First, when the coefficients are constant.

Ex. 1. $\frac{d^2y}{dx^2} + k^2y = 0,$

therefore, putting $y = e^{hx}$, we have, for determining h , the equation $h^2 + k^2 = 0$;

$$\therefore h = \pm k \sqrt{-1};$$

therefore the complete integral will be

$$y = C_1 \cos kx + C_2 \sin kx.$$

Ex. 2. $\frac{d^3y}{dx^3} - \frac{3d^2y}{dx^2} + 4y = 0,$

the equation for determining h is

$$h^3 - 3h^2 + 4 = 0,$$

of which one root is evidently -1 , and dividing by $(h + 1)$, the quotient is $h^2 - 4h + 4 = (h - 2)^2$; therefore the roots of the equation are $-1, 2, 2$; therefore the complete integral is

$$y = ce^{-x} + (a + bx)e^{2x},$$

Ex. 3. $\frac{d^4y}{dx^4} - \frac{2d^3y}{dx^3} + \frac{2d^2y}{dx^2} - \frac{2dy}{dx} + y = 0$

gives, for determining h , the equation

$$h^4 - 2h^3 + 2h^2 - 2h + 1 = 0,$$

which is a recurring equation, evidently having two roots $= +1$, the other two being found to be $+\sqrt{-1}$ and $-\sqrt{-1}$, therefore the complete integral is

$$y = C_1 \cos x + C_2 \sin x + (a + bx)e^x.$$

Ex. 4. $\frac{d^4y}{dx^4} + 2a^2 \frac{d^2y}{dx^2} + a^4 y = 0,$

the equation for h , is $h^4 + 2a^2 h^2 + a^4 = 0,$

$$\text{that is, } (h^2 + a^2)^2 = 0,$$

therefore there is an equal pair of impossible roots, viz.: two roots, each $= a\sqrt{-1}$, and two, each $= -a\sqrt{-1}$; therefore the complete integral, by page 67, is

$$y = \{C_1 + C_2 x\} \cos ax + (C_3 + C_4 x) \sin ax.$$

Secondly, when the coefficients are variable, and of such a form that, by putting $y = e^{\int u dx}$, u can be found.

Ex. 1. $\frac{d^2y}{dx^2} - \frac{2dy}{x dx} + \frac{2y}{x^2} = 0,$

here the coefficients are of the form

$$\frac{A}{(a + bx)}, \text{ and } \frac{B}{(a + bx)^2},$$

therefore assume $y = e^{\int u dx}$, and substituting this and its differential coefficients, we have

$$u^2 + \frac{du}{dx} - \frac{2u}{x} + \frac{2}{x^2} = 0,$$

$$\text{then assume } u = \frac{m}{x},$$

$$\therefore \frac{du}{dx} = -\frac{m}{x^2},$$

therefore, substituting and omitting the common divisor x^2 , we have for determining m , the equation

$$m^2 - 3m + 2 = 0,$$

$$\therefore m^2 - 3m + \frac{2}{1} = \frac{1}{1};$$

$$\therefore m = \frac{3}{2} \pm \frac{1}{2} = 2 \text{ or } 1,$$

$$\therefore u = \frac{2}{x} \text{ or } \frac{1}{x};$$

$$\text{or } \int u dx = \log_e x^2 \text{ or } \log_e x;$$

$$\therefore e^{\int u dx} = x^2, \text{ or } = x;$$

and hence the complete solution of the equation is

$$y = ax^2 + bx.$$

Ex. 2. $\frac{d^2y}{dx^2} + 2bx \frac{dy}{dx} + b^2x^2y = 0,$

then, assuming $y = e^{\int u dx}$, we obtain the equation

$$(u^2 + 2bux + b^2x^2) dx + du = 0,$$

$$\text{or } (u + bx)^2 dx + du = 0;$$

$$\text{then put } u + bx = z,$$

$$\therefore du = dz - bdx,$$

and the equation becomes $z^2 dx + dz - bdx = 0$;

therefore, dividing by $z^2 - b$, we have the integral

$$x = \frac{1}{2\sqrt{b}} \log_e \frac{z + \sqrt{b}}{z - \sqrt{b}} - \frac{1}{2\sqrt{b}} \log_e C_1,$$

where $-\frac{1}{2\sqrt{b}} \log_e C_1$ is taken for the arbitrary constant, for convenience.

From this equation

$$C_1 e^{2x\sqrt{b}} = \frac{z + \sqrt{b}}{z - \sqrt{b}},$$

$$\text{whence } z = \sqrt{b} \cdot \frac{C_1 e^{2x\sqrt{b}} + 1}{C_1 e^{2x\sqrt{b}} - 1};$$

$$\therefore \int u dx = -\int bxdx + \sqrt{b} \int dx \frac{C_1 e^{2x\sqrt{b}} + 1}{C_1 e^{2x\sqrt{b}} - 1},$$

$$\begin{aligned}
&= -\frac{bx^2}{2} + \sqrt{b} \int dx \left\{ \frac{2 C_1 e^{2x\sqrt{b}}}{C_1 e^{2x\sqrt{b}} - 1} - 1 \right\}, \\
&= -\frac{bx^2}{2} + \log_e (C_1 e^{2x\sqrt{b}} - 1) - x\sqrt{b} - \log_e C_2; \\
\therefore y = e^{\int u dx} &= \frac{C_1 e^{2x\sqrt{b}} - 1}{C_2 e^{\left(\frac{bx^2}{2} + x\sqrt{b}\right)}} \\
&= e^{-\frac{bx^2}{2}} \{ c e^{x\sqrt{b}} - c' e^{-x\sqrt{b}} \}.
\end{aligned}$$

Ex. 3. If our equation had been

$$\frac{d^2y}{dx^2} - 2bx \frac{dy}{dx} + b^2x^2y = 0,$$

proceeding in an exactly similar manner, we should find

$$\begin{aligned}
x &= -\frac{1}{\sqrt{b}} \tan^{-1} \frac{z}{\sqrt{b}} - \frac{C_1}{\sqrt{b}}; \\
\therefore x\sqrt{b} + C_1 &= -\tan^{-1} \frac{z}{\sqrt{b}}, \\
\therefore z &= -\sqrt{b} \tan (x\sqrt{b} + C_1), \\
\therefore u &= z + bx, \text{ in this case,} \\
&= bx - \sqrt{b} \tan (x\sqrt{b} + C_1), \\
\int u dx &= \frac{bx^2}{2} + \log_e \{ C_2 \cos (x\sqrt{b} + C_1) \}; \\
\therefore y = e^{\int u dx} &= C_2 \cos (x\sqrt{b} + C_1) e^{\frac{bx^2}{2}} \\
&= e^{\frac{bx^2}{2}} (c \cos x\sqrt{b} - c' \sin x\sqrt{b}).
\end{aligned}$$

To illustrate the method of the variation of parameters, explained in page 62, we will take the equation

$$\frac{d^2y}{dx^2} + A \frac{dy}{dx} + By = X \dots (1),$$

A and B being constant, and X a function of x , its integral, as we have shewn, is to be deduced from the particular integrals of the equation

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0;$$

the equation for determining h , is

$$h^2 + Ah + B = 0;$$

suppose that the values of h are both possible and unequal, and let them be α , β , therefore the particular integrals are $e^{\alpha x}$ and $e^{\beta x}$.

Then we are to assume the integral of equation (1) to be

$$y = ae^{\alpha x} + be^{\beta x},$$

when a and b are functions of x , to be determined;

$$\therefore \frac{dy}{dx} = a\alpha e^{\alpha x} + b\beta e^{\beta x} + a'e^{\alpha x} + b'e^{\beta x},$$

$$\frac{d^2y}{dx^2} = a'\alpha e^{\alpha x} + b'\beta e^{\beta x} + a\alpha' e^{\alpha x} + \beta b'e^{\beta x}$$

+ other terms not involving a' and b' ;

and we assume the equations

$$a'e^{\alpha x} + b'e^{\beta x} = 0,$$

$$a\alpha' e^{\alpha x} + \beta b'e^{\beta x} = X;$$

therefore, eliminating a' , we have

$$b'(\beta - \alpha)e^{\beta x} = X,$$

$$\therefore b' = \frac{X}{(\beta - \alpha)e^{\beta x}};$$

and eliminating b' , we have

$$a' = \frac{X}{(\alpha - \beta)e^{\alpha x}};$$

$$\therefore a = \frac{1}{(\alpha - \beta)} \left(\int \frac{Xdx}{e^{\alpha x}} + C_1 \right),$$

$$b = -\frac{1}{(\alpha - \beta)} \left(\int \frac{Xdx}{e^{\beta x}} + C_2 \right);$$

therefore, the complete integral, in this case, is

$$y = \frac{1}{(\alpha - \beta)} \cdot \left\{ e^{\alpha x} \cdot \left(\int \frac{Xdx}{e^{\alpha x}} + C_1 \right) - e^{\beta x} \left(\int \frac{Xdx}{e^{\beta x}} + C_2 \right) \right\}.$$

If the roots of the equation for h had been both equal to a , the particular integrals would be e^{ax} , and xe^{ax} ; and, therefore, we assume $y = ae^{ax} + bxe^{ax}$, and proceeding as above, we have for determining a' and b' , the equations

$$\begin{aligned} a' + b'x &= 0, \\ b' + (a' + b'x)a &= \frac{X}{e^{ax}}, \end{aligned}$$

whence the complete integral is

$$y = e^{ax} \cdot \left\{ C_1 + C_2 x - \left(\int \frac{X x dx}{e^{ax}} - x \int \frac{X dx}{e^{ax}} \right) \right\}.$$

If the roots be impossible, they must be of the form

$$a + \beta \sqrt{-1}, \text{ and } a - \beta \sqrt{-1},$$

and we assume

$$y = ae^{ax} \cos \beta x + be^{ax} \sin \beta x,$$

whence, differentiating, the equations for a' and b' are

$$\begin{aligned} a' \cos \beta x + b' \sin \beta x &= 0, \\ b' \cos \beta x - a' \sin \beta x &= \frac{X}{\beta e^{ax}}, \end{aligned}$$

and we have for the complete integral

$$\begin{aligned} y &= C_1 e^{ax} \cos \beta x + C_2 e^{ax} \sin \beta x \\ &- \frac{e^{ax}}{\beta} \cdot \left\{ \cos \beta x \int \frac{X \sin \beta x dx}{e^{ax}} - \sin \beta x \int \frac{X \cos \beta x dx}{e^{ax}} \right\}. \end{aligned}$$

The above is the linear equation of the second order, with constant coefficients in its most comprehensive form; particular cases of the equation are to be solved by the same process, without reference to the general results just obtained.

Ex. 1. $\frac{d^2 y}{dx^2} - m^2 y = e^x$, the equation for determining h is $h^2 - m^2 = 0$; therefore, the particular integrals of the equation $\frac{d^2 y}{dx^2} - m^2 y = 0$, are

$$e^{mx}, \text{ and } e^{-mx};$$

$$\therefore \text{assume } y = ae^{mx} + be^{-mx};$$

$$\therefore \frac{dy}{dx} = mae^{mx} - mbe^{-mx} + a'e^{mx} + b'e^{-mx},$$

$$\frac{d^2y}{dx^2} = ma'e^{mx} - mb'e^{-mx} + \text{terms not in-}$$

volving a' and b' ;

therefore, we have the equations

$$a'e^{mx} + b'e^{-mx} = 0,$$

$$a'e^{mx} - b'e^{-mx} = \frac{e^x}{m};$$

$$\therefore a = \frac{e^x}{2me^{mx}} = \frac{1}{2me^{(m-1)x}},$$

$$\text{and } b' = -\frac{e^{(m+1)x}}{2m}.$$

Hence, integrating, we have

$$a = C_1 - \frac{1}{2m(m-1)e^{(m-1)x}},$$

$$b = C_2 - \frac{e^{(m+1)x}}{2m(m-1)};$$

therefore, the complete integral is

$$\begin{aligned} y &= \left\{ C_1 - \frac{1}{2m(m-1)e^{(m-1)x}} \right\} e^{mx} + \left\{ C_2 - \frac{e^{(m+1)x}}{2m(m-1)} \right\} e^{-mx} \\ &= C_1 e^{mx} + C_2 e^{-mx} - \frac{1}{2m} \left\{ \frac{e^x}{(m-1)} + \frac{e^x}{(m+1)} \right\} \\ &= C_1 e^{mx} + C_2 e^{-mx} - \frac{e^x}{(m^2-1)}. \end{aligned}$$

$$\text{Ex. 2. } \frac{d^2y}{dx^2} + k^2y = \cos mx;$$

from this equation we have $h = \pm k\sqrt{-1}$; therefore the par-

ticular integrals of $\frac{d^2y}{dx^2} + k^2y = 0$ are $\cos kx$, and $\sin kx$;

$\therefore$ assume $y = a \cos kx + b \sin kx$,

whence $\frac{dy}{dx} = -ak \sin kx + bk \cos kx + a' \cos kx + b' \sin kx$;

and differentiating again, and collecting the terms involving a' and b' , we have the equations

$$\left. \begin{aligned} a' \cos kx + b' \sin kx &= 0 \\ b' k \cos kx - a' k \sin kx &= \cos mx \end{aligned} \right\},$$

$$\text{or } b' \cos kx - a' \sin kx = \frac{\cos mx}{k},$$

$$b' \sin kx + a' \cos kx = 0;$$

$$\therefore b' (\cos^2 kx + \sin^2 kx) = \frac{\cos kx \cos mx}{k};$$

$$\therefore b' = \frac{\cos kx \cos mx}{k}, \text{ and } a' = -\frac{\sin kx \cos mx}{k},$$

$$\text{or } b' = \frac{1}{k} \cdot \{ \cos (k+m)x + \cos (k-m)x \},$$

$$\text{and } a' = -\frac{1}{k} \cdot \{ \sin (k+m)x + \sin (k-m)x \};$$

$$\therefore b = \frac{1}{2k} \cdot \left\{ \frac{\sin (k+m)x}{(k+m)} + \frac{\sin (k-m)x}{(k-m)} \right\} + C_2,$$

$$a = \frac{1}{2k} \cdot \left\{ \frac{\cos (k+m)x}{(k+m)} + \frac{\cos (k-m)x}{(k-m)} \right\} + C_1;$$

$$\therefore y = C_1 \cos kx + C_2 \sin kx$$

$$+ \frac{1}{2k(k+m)} \{ \cos (k+m)x \cdot \cos kx + \sin (k+m)x \cdot \sin kx \}$$

$$+ \frac{1}{2k(k-m)} \{ \cos (k-m)x \cdot \cos kx + \sin (k-m)x \cdot \sin kx \}$$

$$y = C_1 \cos kx + C_2 \sin kx + \frac{1}{2k} \cdot \left\{ \frac{\cos mx}{k+m} + \frac{\cos (-mx)}{k-m} \right\}$$

$$= C_1 \cos kx + C_2 \sin kx + \frac{\cos mx}{(k^2 - m^2)}.$$

Ex. 3. We will now take an example, in which the equation for h has equal roots,

$$\frac{d^2y}{dx^2} - 2k \frac{dy}{dx} + k^2y = \cos mx$$

gives $(h - k)^2 = 0$;

therefore, the particular integrals of

$$\frac{d^2y}{dx^2} - 2k \frac{dy}{dx} + k^2y = 0$$

are e^{kx} , and xe^{kx} ;

$\therefore$ assume $y = ae^{kx} + bxe^{kx}$;

$$\therefore \frac{dy}{dx} = ake^{kx} + d'e^{kx} + b'xe^{kx} + be^{kx} + b'kxe^{kx},$$

and the terms involving a' and b' , in the value of $\frac{d^2y}{dx^2}$, are

$$a'ke^{kx} + b'e^{kx} + b'kxe^{kx};$$

therefore our equations are

$$a' + b'x = 0,$$

$$ak + b'(1 + kx) = \frac{\cos mx}{e^{kx}};$$

the first equation gives us

$$a' = -b'x,$$

and substituting this in the second, we get

$$-b'kx + b' + b'kx = \frac{\cos mx}{e^{kx}};$$

$$\therefore b' = \frac{\cos mx}{e^{kx}},$$

$$a' = -\frac{x \cos mx}{e^{kx}}.$$

Integrating by parts, we find

$$b = C_2 + \frac{e^{-kx}}{(m^2 + k^2)} \{m \sin mx - k \cos mx\}$$

$$a = C_1 - \frac{e^{-kx}}{m^2 + k^2} \left\{ \left(x + \frac{k}{m^2 + k^2} \right) (m \sin mx - k \cos mx) \right. \\ \left. + \frac{m}{m^2 + k^2} (m \cos mx + k \sin mx) \right\},$$

therefore the complete integral is

$$y = (C_1 + C_2x) e^{kx} - \frac{(m^2 - k^2) \cos mx + 2mk \sin mx}{m^2 + k^2}.$$

Ex. 4. If our equation had been

$$\frac{d^2y}{dx^2} - 2k\frac{dy}{dx} + k^2y = e^x,$$

the integrals would have been much simpler, we should have the assumed integral

$$y = ae^{kx} + bxe^{kx},$$

and our equations for determining the functions a and b , would be

$$a' + b'x = 0,$$

$$ak + b'(1 + kx) = \frac{e^x}{e^{kx}} = e^{-(k-1)x};$$

whence, eliminating as before,

$$b' = e^{-(k-1)x};$$

$$\therefore b = C_2 - \frac{1}{(k-1)} e^{-(k-1)x},$$

$$a' = -xe^{-(k-1)x};$$

$$\therefore a = C_1 + \frac{(k-1)x + 1}{(k-1)^2} e^{-(k-1)x};$$

therefore the complete integral is

$$y = (C_1 + C_2x) e^{kx} + \frac{e^x}{(k-1)^2}.$$

The general method is easily applied to the solution of the equation

$$\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \frac{2y}{x^2} = X;$$

we have seen (Ex. 1, page 71) that the integral of the equation without X is $y = ax^2 + bx$, where a and b are arbitrary constants,

$\therefore$ assume $y = ax^2 + bx$, a and b functions of x ;

$$\therefore \frac{dy}{dx} = 2ax + b + a'x^2 + b'x,$$

$$\frac{d^2y}{dx^2} = 2a'x + b' + \text{terms not involving } a' \text{ and } b';$$

therefore, for the determination of a' , b' , we have the assumed equations

$$\begin{aligned} a'x + b' &= 0, \\ 2a'x + b' &= X; \\ \therefore a' &= \frac{X}{x}, \therefore a = C_1 \int Xx^{-1}dx, \\ b' &= -X, \therefore b = C_2 - \int Xdx; \end{aligned}$$

therefore the required integral is

$$y = C_1x^2 + C_2x + x^2 \int Xx^{-1}dx - x \int Xdx,$$

the constants of the integrations being C_1 and C_2 respectively.

SECT. 3. *On particular artifices and methods of solution applicable to particular forms of linear Equations.*

The methods above exemplified are certainly the most general that are known; at the same time there are others which, in certain cases, lead more directly and easily to the required solutions. There are also transformations, which enable us to integrate some linear equations, which, without them, could not be solved by any known process.

The analyst cannot be in possession of too many ways of treating equations, of which there are such a variety of forms; and we therefore devote this section to the explanation of the methods above alluded to.

The equation $\frac{d^2y}{dx^2} + A \frac{dy}{dx} + By = X$, (the coefficients being constant) may be solved without applying the variation of parameters.

For convenience, represent the equation thus

$$y'' + Ay' + By = X,$$

and suppose $e^{-hx}dx$ a factor which will make it immediately integrable, so as to give a primitive which shall be a differential equation of the first order;

$$\text{then } e^{-hx}dx (y'' + Ay' + By) = e^{-hx}dxX, \dots (1),$$

and this being a complete differential, assume the integral to be

$$e^{-hx} (ay + by') = (P + C);$$

then, differentiating, this equation

$$-he^{-hx} dx (ay + by') + e^{-hx} dx (ay' + by'') = e^{-hx} dx X,$$

should be identical with equation (1), therefore, equating like coefficients,

$$\begin{aligned} b &= 1, \\ a - bh &= A, \\ -ah &= B, \end{aligned}$$

hence $a = (A + h)$, and two values of h will be found from the equation

$$B + (A + h)h = 0, \text{ or } h^2 + Ah + B = 0;$$

suppose these two values to be α and β , then respecting corresponding to these values, we shall have the primitives

$$e^{-\alpha x} \cdot \{(A + \alpha)y + y'\} = (P + C_1) \dots \dots \dots \}$$

$$\text{and } e^{-\beta x} \cdot \{(A + \beta)y + y'\} = (P + C_2) \dots \dots \dots \} \dots \dots (2),$$

and eliminating y' between them, we shall have the complete primitive of the given equation.

A similar process may be applied to equations of higher orders.

For a simple example,

$$\text{let } \frac{d^2y}{dx^2} - k^2y = e^{mx},$$

the values of h are $\pm k$, and $A = 0$,

$$P = \int e^{(m-k)x} dx = \frac{e^{(m-k)x}}{m-k},$$

therefore the equations (2) become

$$e^{-kx} \cdot \{ky + y'\} = \frac{e^{(m-k)x}}{m-k} + C_1,$$

$$e^{kx} \cdot \{-ky + y'\} = \frac{e^{(m+k)x}}{m+k} + C_2;$$

therefore, multiplying the first by e^{kx} and the second by e^{-kx} , and subtracting the second from the first, we get

$$2ky = \frac{2me^{mx}}{m^2 - k^2} + C_1 e^{kx} - C_2 e^{-kx}$$

for the complete integral.

There is a method of integrating the equation

$$\frac{d^2y}{dx^2} + k^2y + X = 0,$$

which applies so conveniently to the investigations of physical astronomy, that we must not omit it.

Multiply the equation by $\cos kx$

$$\therefore \cos kx \frac{d^2y}{dx^2} + k^2y \cos kx + X \cos kx = 0,$$

$$\therefore \int \cos kx \frac{d^2y}{dx^2} dx + \int k^2y \cos kx dx + \int X \cos kx dx = C_1,$$

therefore, integrating the first and second terms by parts,

$$\frac{dy}{dx} \cos kx + \int k \frac{dy}{dx} \sin kx dx + ky \sin kx - \int k \frac{dy}{dx} \sin kx dx + \int X \cos kx dx = C_1$$

$$\text{or } \frac{dy}{dx} \cos kx + ky \sin kx = C_1 - \int X \cos kx dx \dots (1).$$

Then, if we multiply the given equation by $\sin kx$, and proceed to integrate by parts, we shall find

$$\frac{dy}{dx} \sin kx - ky \cos kx = C_2 - \int X \sin kx dx \dots (2);$$

therefore, multiplying equation (1) by $\sin kx$, and (2) by $\cos kx$, and subtracting the latter product from the former, we shall have

$$ky (\sin^2 kx + \cos^2 kx) = \sin kx \cdot (C_1 - \int X \cos kx dx) - \cos kx \cdot (C_2 - \int X \sin kx dx)$$

$$\text{or } y = \frac{\sin kx}{k} (C_1 - \int X \cos kx dx) - \frac{\cos kx}{k} (C_2 - \int X \sin kx dx),$$

which is the complete integral of the proposed equation.

Ex. Suppose the value of X in the equation to have been $\cos^2 kx$, then

$$\int X \cos kx dx = \int (1 - \sin^2 kx) \cos kx dx,$$

$$= \frac{1}{k} \left\{ \sin kx - \frac{\sin^3 kx}{3} \right\},$$

$$\int X \sin kx dx = -\frac{\cos^3 kx}{3k},$$

$$\therefore y = \frac{1}{k} \cdot \left\{ (C_1 - 1) \sin kx - C_2 \cos kx + \frac{1}{3} (\sin^3 kx + \cos^3 kx) \right\}.$$

or, since C_1 is arbitrary, therefore $C_1 - 1$ is arbitrary; and therefore the equation may be written

$$y = \frac{1}{k} \left\{ A \sin kx - B \cos kx + \frac{\sin^3 kx + \cos^3 kx}{3} \right\}.$$

The equation $\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = 0$, in the case in which

$Q = -\frac{P}{x}$ may generally be solved by a simple transformation.

Assume $y = zx$, $\therefore \frac{dy}{dx} = z + x \frac{dz}{dx}$,

$$\text{and } \frac{d^2y}{dx^2} = 2 \frac{dz}{dx} + x \frac{d^2z}{dx^2}.$$

Substituting these values and dividing by x , we have the equation

$$\frac{d^2z}{dx^2} + \frac{2}{x} \frac{dz}{dx} + P \frac{z}{x} + P \frac{dz}{dx} + Qz = 0,$$

$$\text{or, since } Q = -\frac{P}{x},$$

$$\frac{d^2z}{dx^2} + \frac{dz}{dx} \cdot \left(\frac{2}{x} + P \right) = 0;$$

which equation, as we have before shewn, may be easily solved.

$$\text{Ex. } \frac{d^2y}{dx^2} + \frac{3x-2}{x(1-x)} \frac{dy}{dx} - \frac{3x-2}{x^2(1-x)} y = 0,$$

the transformed equation becomes

$$\frac{d^2z}{dx^2} + \frac{1}{1-x} \frac{dz}{dx} = 0,$$

and putting $\frac{dz}{dx} = p$, we obtain

$$\log_e p = \log_e (1-x) + \log_e C_1, \therefore \frac{dz}{dx} = C_1(1-x),$$

$$\text{whence } z = C_1 \left(x - \frac{x^2}{2} \right) + C_2;$$

$$\therefore y = C_1 \left(x^2 - \frac{x^3}{2} \right) + C_2 x,$$

The transformation of the independent variable sometimes reduces the equation to a more simple form.

Suppose our equation of the form

$$\frac{d^2y}{dx^2} + \frac{A + 2x}{1 + x^2} \frac{dy}{dx} + \frac{By}{(1 + x^2)^2} = 0,$$

when A and B are constants.

Then transform the equation so that $\theta = \tan^{-1} x$ shall be the independent variable. We easily find, by the methods of the Differential Calculus, that

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{d\theta} \cos^2 \theta, \\ \frac{d^2y}{dx^2} &= \frac{\frac{d^2y}{d\theta^2} - 2 \frac{dy}{d\theta} \tan \theta}{\sec^4 \theta}, \end{aligned}$$

therefore substituting, our equation becomes

$$\frac{\frac{d^2y}{d\theta^2} - 2 \frac{dy}{d\theta} \tan \theta}{\sec^4 \theta} + \frac{A + 2x}{1 + x^2} \frac{dy}{d\theta} \cos^2 \theta + \frac{By}{(1 + x^2)^2} = 0,$$

or observing that $\tan \theta = x$, $\therefore \sec^2 \theta = 1 + x^2$, we get

$$\frac{d^2y}{d\theta^2} + A \frac{dy}{d\theta} + By = 0;$$

in which, the coefficients being constant, we shall be able to integrate, and find y in terms of θ ; that is, y in terms of $\tan^{-1} x$.

In a similar manner the equation

$$\frac{d^2y}{dx^2} + \frac{A\sqrt{2x-x^2} + 1-x}{2x-x^2} \frac{dy}{dx} + \frac{By}{2x-x^2} = 0$$

is reduced to the form

$$\frac{d^2y}{d\theta^2} + A \frac{dy}{d\theta} + By = 0,$$

by transforming it to one in which $\theta = \text{vers}^{-1} x$ is the independent variable.

Again, the equation

$$\frac{d^2y}{dx^2} + \frac{A\sqrt{1-x^2}-x}{1-x^2} \frac{dy}{dx} + \frac{By}{1-x^2} = 0$$

is reduced to the same form, by making $\theta = \sin^{-1} x$ or $\cos^{-1} x$, the independent variable.

$$\text{Also, } \frac{d^2y}{dx^2} + (Ae^x - 1) \frac{dy}{dx} + Bye^{2x} = 0$$

$$\text{becomes } \frac{d^2y}{dv^2} + A \frac{dy}{dv} + By = 0$$

by making $v = e^x$, the independent variable.

It is needless in this place to give particular examples of the above, as the method of proceeding is precisely the same as that in the first case. Should either of the quantities A or B be wanting, the transformed equation will of course be without the term in which A or B respectively enters.

Some equations are made integrable by transforming them so that y shall be the independent variable instead of x .

$$\text{Ex. } a^4xy^2 \left(\frac{dy}{dx}\right) + 2a^2y \left(\frac{dy}{dx}\right)^2 - \frac{d^2y}{dx^2} = 0$$

is, by the above means, reduced to a linear equation in x ;

$$\text{for } \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}; \quad \frac{d^2y}{dx^2} = -\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}$$

therefore, substituting these values, our equation becomes

$$\frac{d^2x}{dy^2} + 2a^2y \frac{dx}{dy} + a^4y^2x = 0;$$

then assuming $x = e^{\int u dy}$, and substituting this quantity and its differential coefficients,

$$u^2 + 2a^2yu + a^4y^2 + \frac{du}{dy} = 0,$$

$$\text{or } (u + a^2y)^2 + \frac{du}{dy} = 0,$$

$$\text{put } u + a^2y = z, \therefore \frac{du}{dy} = \frac{dz}{dy} - a^2,$$

and we have

$$dy = \frac{dz}{a^2 - z^2},$$

$$\therefore y = \frac{1}{2a} \log_e \left(\frac{1}{C_1} \frac{a+z}{a-z} \right),$$

$$\therefore C_1 e^{2ay} = \frac{a+z}{a-z},$$

$$\therefore z = a \frac{C_1 e^{2ay} - 1}{C_1 e^{2ay} + 1},$$

$$\text{or } u = a \frac{C_1 e^{2ay} - 1}{C_1 e^{2ay} + 1} - a^2 y,$$

$$\therefore \int u dy = \log_e (C_1 e^{2ay} + 1) - ay - \frac{a^2 y^2}{2} - \log_e C_1,$$

$$\therefore x = e^{\int u dy} = \frac{C_1 e^{2ay} + 1}{C_2 e^{(ay + \frac{a^2 y^2}{2})}}$$
 is the complete integral of

the given equation.

There are many equations of the second order, which being neither linear nor always of the first degree, no general methods can be given for their solution.

The integration, when possible, is generally effected through the medium of substitutions and transformations, which practice only can teach; we have, therefore, introduced some examples of this sort in the small collection at the end of this volume, where we have also indicated the method of proceeding to be applied to each.

Several complicated equations (not linear) are integrated by assuming $y = e^{\int u dx}$, of which the following is an example:

$$y \frac{d^2 y}{dx^2} + P \left(\frac{dy}{dx} \right)^2 + Q y \frac{dy}{dx} = 0,$$

where P and Q are functions of x.

$$\text{Assume } y = e^{\int u dx}, \therefore \frac{dy}{dx} = u e^{\int u dx},$$

$$\text{and } \frac{d^2 y}{dx^2} = \frac{du}{dx} e^{\int u dx} + u^2 e^{\int u dx};$$

then substituting these values, and omitting the common factor $e^{2\int u dx}$, we have

$$\frac{du}{dx} + (P + 1) u^2 + Qu = 0.$$

Then assuming $u = \frac{1}{v}$,

$$\therefore \frac{du}{dx} = -\frac{1}{v^2} \frac{dv}{dx},$$

and substituting, we have the equation

$$dv - Qv dx = (P + 1) dx,$$

which is a linear equation for determining v , and therefore u will be found; and in the value of u a constant will be introduced, and in the second integration for $\int u dx$ another constant will occur, therefore the value $y = e^{\int u dx}$, thus obtained, will be the complete solution.

It is easily perceived that the substitution of $y = e^{\int u dx}$ will only be available when the equation is homogeneous with respect to y and its differential coefficients.

SECT. 4. *On the singular solutions of equations of superior orders.*

Much of what has been said respecting singular solutions of equations of the first order, applies also to those of superior orders.

Represent the differential coefficients

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n} \text{ by } y', y'', \dots, y^{(n)};$$

and suppose $\phi(x, y, y', y'', \dots, y^{(n)}) = U_{(n)} = 0$ to be a differential equation of the n^{th} order, and

$$f(x, y, y', y'', \dots, y^{(n-1)}, a) = U_{(n-1)} = 0$$

one of the first integrals, which is a differential equation of the $(n-1)^{\text{th}}$ order, involving an arbitrary constant a .

If we differentiate $U_{(n-1)}$ with respect to a , and, assuming the equation $\frac{dU_{(n-1)}}{da} = 0$, find a value of a in terms of

$x, y, y', \dots y^{(n-1)}$, this value substituted in $U_{(n-1)}$ gives us a singular solution of the equation $U_{(n)} = 0$.

Ex. The equation $(y')^2 - \frac{2y y''}{x} + 1 = 0$ has for its primitive of the first order

$$x^2 - 2ay' + a^2 = 0 \dots (1).$$

Then differentiating this with respect to a , and assuming the result to equal nothing, we get $a = y^2$, which value substituted in (1), gives the singular solution $y' = \pm x$.

It has been demonstrated by *Lagrange*, that generally the n first integrals of a differential equation of the n^{th} order, all give the same singular solution; we shall shew that this is the case in the above example.

Integrating $x^2 - 2ay' + a^2 = 0$, we have

$$\frac{x^3}{3} - 2ay + a^2x + b = 0,$$

where b is the new arbitrary constant, and eliminating a by means of these two equations, we have

$$4x^2(y - xy')^2 - 4\left(b - \frac{2x^3}{3}\right)(y - xy')y' + \left(b - \frac{2x^3}{3}\right)^2 = 0 \dots (2),$$

which is therefore the remaining integral of the first order of the differential equation.

Differentiating with respect to b ,

$$-4(y - xy')y' + 2\left(b - \frac{2x^3}{3}\right) = 0,$$

whence the value of b is

$$\frac{2x^3}{3} + 2(y - xy')y';$$

and this value, substituted in (2), gives

$$4(y - xy')^2(x^2 - y'^2) = 0.$$

Now the supposition that $y - xy' = 0$ does not satisfy the proposed equation, therefore we must have

$$x^2 - (y')^2 = 0, \text{ or } y' = \pm x,$$

which is the same singular solution as was obtained by the direct treatment of equation (1).

If $U_{(n-1)}$ be the integral of $U_{(n)} = 0$, involving the arbitrary constant a ; then, if the value of a be found from the equation $U_{(n-1)} = 0$, the conditions that make

$$\frac{da}{dx}, \frac{da}{dy}, \frac{da}{dy'}, \frac{da}{dy''} \dots$$

all infinite, may give us a singular solution of the equation $U_{(n)} = 0$.

For, differentiating $U_{(n-1)} = 0$ considering a as variable, and a function of $x, y, y', y'', \dots$ we have

$$\frac{dU_{(n-1)}}{dx} + \frac{dU_{(n-1)}}{da} \frac{da}{dx} = 0,$$

$$\frac{dU_{(n-1)}}{dy} + \frac{dU_{(n-1)}}{da} \frac{da}{dy} = 0,$$

$$\frac{dU_{(n-1)}}{dy'} + \frac{dU_{(n-1)}}{da} \frac{da}{dy'} = 0,$$

$$\dots + \dots = 0;$$

$$\therefore \frac{da}{dx} = - \frac{\left(\frac{dU_{(n-1)}}{dx} \right)}{\left(\frac{dU_{(n-1)}}{da} \right)}; \quad \frac{da}{dy} = - \frac{\left(\frac{dU_{(n-1)}}{dy} \right)}{\left(\frac{dU_{(n-1)}}{da} \right)}$$

$$\frac{da}{dy'} = - \frac{\left(\frac{dU_{(n-1)}}{dy'} \right)}{\left(\frac{dU_{(n-1)}}{da} \right)}, \dots = \dots$$

But the denominators in all these expressions will equal zero if the value of a be such as will give a singular solution; and, therefore, assuming the equations

$$\frac{da}{dx} = \infty; \quad \frac{da}{dy} = \infty, \dots$$

the proper relation may sometimes be determined.

We should observe, that although the singular solution makes the above quantities infinite, it does not follow that *any* solution which does this is a singular solution; for if it reduce the value of a (found from the equation $\frac{dU_{(n-1)}}{da} = 0$) to a constant, it will only be a particular integral.

If $V_{(n)} = 0$ be a differential equation of the n^{th} order, which requires a factor μ to make it the complete differential of $U_{(n-1)} = 0$; and if $V_{(n-1)} = 0$ be a singular solution of the equation $V_{(n)} = 0$, then the supposition $V_{(n-1)} = 0$ makes the factor μ infinite.

For the solution $V_{(n-1)} = 0$ is not included in the equation $U_{(n-1)} = 0$, because the former is a singular solution, and, therefore, the relation expressed by $V_{(n-1)} = 0$ will not make $U_{(n)} = 0$.

But it does make $V_{(n)}$ or $U_{(n)} \cdot \frac{1}{\mu} = 0$, and, therefore, it must make $\mu = \infty$.

To obtain a singular solution from the differential equation, without previously knowing its complete integral.

Suppose $F(x, y, y', \dots y^{(n-1)}, a) = 0$ to be the integral of a differential equation of the n^{th} order. Then, differentiating this with respect to x , a value of a in terms of $x, y, y', \dots y^{(n)} = P$, suppose, might be obtained, which, substituted in the above, would give $F(x, y, y', \dots y^{(n-1)}, P) = 0$; an equation identical with the given differential equation of the n^{th} order.

But if this last be differentiated, the terms not depending on the variation of P would be identically nothing; and, therefore, the result of differentiating our equation of the n^{th} order, must be of the form

$$\frac{d(F, x, y, y', \dots y^{(n-1)}, P)}{dP} \cdot \frac{dP}{dx} = 0;$$

that is, the result will, when our method applies, be divisible into two factors respectively of the n^{th} and $(n+1)^{\text{th}}$ orders.

The factor of the $(n+1)^{\text{th}}$ order, $\frac{dP}{dx}$ gives an integral of the n^{th} order, $P = a$, and, from this, finding the value of $y^{(n)}$, and substituting in the given equation, we should have a complete integral of it.

The other factor of the n^{th} order equated to zero, will enable us to eliminate $y^{(n)}$ from the given equation, thus obtaining a relation between $x, y, y', \dots, y^{(n-1)}$, which does not involve a constant, and is the singular solution required.

It will be seen that the process is analogous to that employed to obtain singular solutions of those equations of the first order, which come under the form $y = px + P$, treated of in page 44.

Ex. The equation $(y'')^2 - \frac{2y'y''}{x} + 1 = 0$, gives, by differentiation,

$$2\left(y'' - \frac{y'}{x}\right)y''' - \frac{2(y'')^2}{x} + \frac{2y'y''}{x^2} = 0,$$

which may be put in the form

$$2\left(y'' - \frac{y'}{x}\right)\left(y''' - \frac{y''}{x}\right) = 0.$$

The first factor, which is of the second order, equated to zero, gives $y'' = \frac{y'}{x}$, and substituting this value in the given equation, we have the singular solution

$$-\frac{(y')^2}{x^2} + 1 = 0, \text{ or } x^2 - (y')^2 = 0,$$

which is the same as we have previously obtained.

The other factor gives $\frac{y''}{x} = \frac{1}{a^2} \therefore y'' = \frac{x}{a^2}$, and substituting this value in the given equation, the result is

$$x^2 - 2ay' + a^2 = 0,$$

which is the complete primitive of the first order.

CHAPTER VI.

ON THE SOLUTION OF SIMULTANEOUS EQUATIONS.

WHEN we have two or more differential equations between three or more variables, which are to be all satisfied by the same values of these variables, these equations are called simultaneous. The investigations connected with such equations properly belong to this *part*, because, as will be seen, the *integrations* to be performed are only those of equations between two variables.

We shall proceed at once to the solution of some of the most useful forms.

First, let $(ax + by) dt + dx = 0 \dots \dots (1),$

$(a'x + b'y) dt + dy = 0 \dots \dots (2).$

Multiply the second equation by an indeterminate quantity k , and then adding the product to the first, we have

$\{(a + ka')x + (b + kb')y\} dt + dx + k dy = 0 \dots (3),$

which will be integrable, if $dx + k dy$ multiplied by some constant is the differential of the quantity multiplying dt ; therefore, taking h , a second indeterminate quantity, assume

$h(x + ky) = (a + ka')x + (b + kb')y \dots (4),$

whence, equating the coefficients of x and y , we have

$h = a + ka'; \quad hk = b + kb',$

and from these we are to find k and h in terms of $a, b, a', b'.$

Since the equations for k and h will be of two dimensions, each of these quantities will have two values, which we will represent by k, k' , and h, h' .

Then let (5) $\dots x + ky = u; x + k'y = u' \dots$ (6);
and $\therefore$ (7) $\dots dx + kdy = du, dx + k'dy = du' \dots$ (8),
if we make in (3) the substitutions (4), (5), and (7); (4), (6), and (8), we shall have the equations

(9) $\dots h u dt + du = 0; h' u' dt + du' = 0 \dots$ (10),
the integrals of which are

$$u = C e^{-ht}, u' = C_1 e^{-h't}.$$

Substituting these values of u and u' in (5) and (6), we shall have

$$x + ky = C e^{-ht}, x + k'y = C_1 e^{-h't},$$

whence

$$x = \frac{k C_1 e^{-h't} - k' C e^{-ht}}{k - k'} \dots \dots (11),$$

$$y = \frac{C e^{-ht} - C_1 e^{-h't}}{k - k'} \dots \dots (12).$$

Then x and y being expressed both in terms of t , corresponding values of these for any given value of t may be determined, or, indeed, t may be eliminated.

When $k = k'$ and $h = h'$, these values of x and y become infinite; but, in this case, the equations (5) and (6) are reduced to one, which is

$$x + ky = u;$$

also, the equations (9) and (10) become

$$h u dt + du = 0,$$

which gives

$$u = C e^{-ht} = x + ky;
\therefore x = C e^{-ht} - ky \dots \dots (13).$$

Then substituting this value of x in equation (2), and putting $b' - a'k = l$, we have

$$dy + l y dt + a' C e^{-ht} dt = 0,$$

the integral of which is

$$ye^{lt} + \frac{d(Ce^{(l-h)t})}{l-h} = C_1 \dots (14),$$

and the equations (13), (14) give x and y in terms of t .

Again, if the values of k and h be impossible, suppose

$$h \text{ or } h' = \rho \pm \theta\sqrt{-1},$$

$$k \text{ or } k' = a \pm \beta\sqrt{-1}.$$

Then, substituting in (11) and (12), and making use of the exponential forms for sine and cosine as we did in page 64, for the linear equation, we get

$$x = e^{-\rho t} \cdot \left\{ \left(C' - C' \frac{a}{2\beta} \right) \cos \theta t + \left(C' + C' \frac{a}{2\beta} \right) \sin \theta t \right\},$$

$$y = e^{-\rho t} \cdot \left\{ \frac{C'}{2\beta} \cos \theta t - \frac{C''}{2\beta} \sin \theta t \right\},$$

$$\text{where } C' = (C_1 - C)\sqrt{-1},$$

$$\text{and } C'' = (C_1 + C).$$

Secondly, to solve the equations

$$\left. \begin{aligned} (az + by) dx + cdz + edy &= X_1 dx \\ (hz + iy) dx + kdz + ldy &= X_2 dx \end{aligned} \right\} \dots (1),$$

$a, b, c, d, e, h, i, k, l$ being constant, and X_1, X_2 functions of x .

Eliminating successively dy and dz between the two equations, we get

$$\begin{aligned} \{ (al - eh)z + (bl - ei)y \} dx + (cl - ek) dz &= (lX_1 - eX_2) dx, \\ \{ (ch - ak)z + (ci - bk)y \} dx + (cl - ek) dk &= (cX_1 - kX_2) dx; \end{aligned}$$

if we divide by $cl - ek$ (and for conciseness put

$$\frac{al - eh}{cl - ek} = a', \quad \frac{bl - ei}{cl - ek} = b', \quad \frac{lX_1 - eX_2}{cl - ek} = X_{11},$$

in the first equation, and the similar quantities equal a'', b'', X_{21} , in the second) the two become

$$(a'z + b'y) dx + dz = X_{11} dx,$$

$$(a''z + b''y) dx + dy = X_{21} dx.$$

Then, to solve these, multiply the first by an indeterminate factor β , and add the product to the second, and we have

$$\{(\alpha'\beta + \alpha'')z + (\beta'\beta + \beta'')y\}dx + \beta dz + dy = (\beta X_1 + X_2)dx,$$

which becomes

$$(\beta'\beta + \beta'') \cdot \left\{ \frac{\alpha'\beta + \alpha''}{\beta'\beta + \beta''} z + y \right\} dx + \beta dz + dy = (\beta X_1 + X_2) dx, \dots (2),$$

then β being indeterminate, we may suppose

$$\frac{\alpha'\beta + \alpha''}{\beta'\beta + \beta''} = \beta,$$

which will give us a quadratic equation for determining two values of β ; viz: β' β'' , which values successively substituted (2) will give us the two equations

$$(\beta'\beta' + \beta'')(\beta'z + y)dx + \beta' dz + dy = (\beta'X_1 + X_2)dx,$$

$$(\beta'\beta'' + \beta'')(\beta''z + y)dx + \beta'' dz + dy = (\beta''X_1 + X_2)dx;$$

$$\text{which, putting } \beta'\beta' + \beta'' = p',$$

$$\beta'\beta'' + \beta'' = p'',$$

$$\beta'z + y = u', \quad \beta''z + y = u''; \dots (3),$$

$$\beta'X_1 + X_2 = X_1, \quad \beta''X_1 + X_2 = X_2,$$

become

$$p'u'dx + du' = X_1 dx, \quad p''u''dx + pu'' = X_2 dx,$$

which are linear equations of the first order, from which u' and u'' must be found in terms of x and y , and from equations (3) we have

$$y = \frac{\beta'u'' - \beta''u'}{\beta' - \beta''}, \quad z = \frac{u' - u''}{\beta' - \beta''};$$

therefore, substituting the values of u' and u'' , we get two equations between the quantities x , y , z .

Thirdly, to integrate the equations

$$(Mz + Nx)dt + Pdz + Qdx = Tdt \quad \left. \begin{array}{l} \dots (1), \\ (M'z + N'x)dt + P'dz + Q'dy = T'dt \end{array} \right\} \dots (2),$$

when all the coefficients are functions of the independent variable t .

Multiply the second by θ an indeterminate function of t , and adding the product to the first, we get

$$\begin{aligned} \{ (M + M'\theta)z + (N + N'\theta)x \} dt + (P + P'\theta)dx + (Q + Q'\theta)dz \\ = (T + T'\theta)dt, \\ \text{or } (M + M'\theta) \cdot \left\{ z + \frac{N + N'\theta}{M + M'\theta} \right\} dt + P + P'\theta \cdot \left\{ dz + \frac{Q + Q'\theta}{P + P'\theta} dx \right\} \\ = (T + T'\theta)dt \dots (3), \end{aligned}$$

which equation would be of the first order between two variables if we had

$$dz + \frac{Q + Q'\theta}{P + P'\theta} dx = d \cdot \left\{ z + \frac{N + N'\theta}{M + M'\theta} \right\} \dots (4);$$

$$\text{because, putting } z + \frac{N + N'\theta}{M + M'\theta} x = u \dots (5),$$

equation (3) would, after dividing by $P + P'\theta$, become

$$du + \frac{M + M'\theta}{P + P'\theta} u dt = \frac{T + T'\theta}{P + P'\theta} dt \dots (6).$$

This is a linear equation of the first order, and may be integrated.

But in order that the relation (4) may hold, we must have, at the same time,

$$\frac{Q + Q'\theta}{P + P'\theta} = \frac{N + N'\theta}{M + M'\theta}, \text{ and } d \cdot \left(\frac{N + N'\theta}{M + M'\theta} \right) = 0;$$

therefore, eliminating θ from the second of these equations by means of the first, there will remain a relation between the coefficients $M, N, P, Q, M', N', P', Q'$. But when the coefficients are constant, the second equation is satisfied by taking $\theta = \text{constant}$, and the first equation will determine two values of θ . Therefore, representing

the two values of θ by θ_1, θ_2 ,

$$\dots \text{ of } \frac{M + M'\theta}{P + P'\theta} \text{ by } m_1, m_2,$$

$$\dots \text{ of } \frac{N + N'\theta}{M + M'\theta} \text{ by } n_1, n_2,$$

$$\dots \text{ of } \frac{T + T'\theta}{P + P'\theta} \text{ by } T_1, T_2,$$

and substituting these values in equations (5) and (6), and obtaining the two corresponding integrals from equation (6), we shall have to eliminate t .

Fourthly, simultaneous equations, in which the functions of the independent variable and their differential coefficients amount to the same dimensions in every term, are easily solved by assuming $x = e^{mt}$, $y = ae^{mt} \dots$, (t being the independent variable), and substituting in the equation, when the whole will be divisible by e^{mt} , and we shall have algebraic equations for determining m , a , $\dots$.

$$\begin{array}{l} \text{Ex. } \frac{2d^2x}{dt^2} + \frac{dx}{dt} + x + \frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 0 \\ \frac{d^2x}{dt^2} + 96\frac{dx}{dt} - 9x + \frac{d^2y}{dt^2} + 50\frac{dy}{dt} + 15y = 0 \end{array} \quad \left. \vphantom{\begin{array}{l} \text{Ex. } \frac{2d^2x}{dt^2} + \frac{dx}{dt} + x + \frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 0 \\ \frac{d^2x}{dt^2} + 96\frac{dx}{dt} - 9x + \frac{d^2y}{dt^2} + 50\frac{dy}{dt} + 15y = 0 \end{array}} \right\}$$

then substituting e^{mt} and its differential coefficients for x , and its differential coefficients, and $ae^{mt} \dots$ for y , $\dots$ we get for determining m and a , the equations

$$2m^2 + m + 1 + a(m^2 + m + 1) = 0,$$

$$m^2 + 96m - 9 + a(m^2 + 50m + 15) = 0.$$

Eliminating a we get for m the equation

$$m^4 + 4m^3 - 7m^2 - 22m + 24 = 0.$$

The values of m are 1, 2, -3, -4, and the corresponding values of a , are $-\frac{4}{3}$, $-\frac{1}{7}$, $-\frac{1}{7}$, $-\frac{2}{13}$;

$$\therefore x = C_1 e^t + C_2 e^{2t} + C_3 e^{-3t} + C_4 e^{-4t};$$

$$\text{and } y = -\frac{4}{3} C_1 e^t - \frac{1}{7} C_2 e^{2t} - \frac{1}{7} C_3 e^{-3t} - \frac{2}{13} C_4 e^{-4t}.$$

From these t is to be eliminated.

CHAPTER VII.

ON THE APPROXIMATE SOLUTION OF DIFFERENTIAL EQUATIONS.

WHEN other means do not enable us to find the integral of a given equation, we must have recourse to some method of approximation.

The first which presents itself, is deduced from *Taylor's* theorem.

Supposing $y = f(x)$, the integral required, we have, by *Taylor's* theorem,

$$f(x + h) = fx + \frac{dy}{dx} h + \frac{d^2y}{dx^2} \frac{h^2}{1.2} + \frac{d^3y}{dx^3} \frac{h^3}{1.2.3} \dots$$

$$\therefore f(a + h) = A + A' h + A'' \frac{h^2}{1.2} + A''' \frac{h^3}{1.2.3} \dots$$

representing by $A, A' \dots$, the values of $fx, \frac{dfx}{dx}, \dots$ for the value (a) of x .

Then, if $h = (x - a)$ this series gives us

$$fx = y = A + A' (x - a) + A'' \frac{(x - a)^2}{1.2} + A''' \frac{(x - a)^3}{1.2.3} + \dots$$

then, as we have shewn in page 4, as many of the quantities $A, A' \dots$ will be arbitrary as there are units in the order of the equation, and the other coefficients may be determined, so that we shall have a series for y in terms of x , which will give

us the value of y with great accuracy for any value of x nearly equal to a , and since the value of a may be any we please, a value of y may be determined corresponding to any given value of x .

Ex. Let $\frac{dy}{dx} = mx^n - y$, $\therefore A' = \{ma^n - A\}$,

$$\therefore \frac{d^2y}{dx^2} = mn x^{n-1} - \frac{dy}{dx}, \therefore A'' = \{mna^{n-1} - ma^n + A\},$$

$$\frac{d^3y}{dx^3} = mn(n-1)x^{n-2} - \frac{d^2y}{dx^2},$$

$$\therefore A''' = \{mn(n-1)a^{n-2} - mna^{n-1} + ma^n - A\},$$

$$\dots = \dots$$

$$\therefore y = A + (ma^n - A)(x-a) + (mna^{n-1} - ma^n + A)\frac{(x-a)^2}{1.2}$$

$$+ \{mn(n-1)a^{n-2} - mna^{n-1} + ma^n - A\}\frac{(x-a)^3}{1.2.3} + \dots$$

is the series for approximating to the value of y .

The above process sometimes fails, in consequence of a failure of *Taylor's* theorem, for the particular value (a) of x ; in this case we must proceed by the method of indeterminate coefficients, as follows.

Put $(z+a)$ for x , and $(u+b)$ for y , in the given equation; then, if from the equation thus transformed, we can find a series for u in terms of z , such that $u=0$, when $z=0$, and substitute in the expansion $x-a$ for z , at the same time putting $y-b$ for u , this expansion will enable us to approximate to a value of y ; and since it contains the arbitrary constant b , this value will be the complete integral.

To obtain the expansion, we must assume

$$u = Az^\alpha + Bz^\beta + Cz^\gamma + \dots$$

when differentiating this series, and substituting the values of the differential coefficients in the given equation, we shall have an expression which we may assume identical with the

equation, and from this condition the quantities $A, B, C \dots$, $\alpha, \beta, \gamma, \dots$ will be determined.

If we had assumed a series $y = Ax^\alpha + Bx^\beta + \dots$ at once, without making any transformation, we might find a series for y in terms of x , but not involving any arbitrary constant it would not be the complete integral.

Ex. Let $dy + ydx = mx^n dx$,

therefore, putting $(n + b)$ for y , and $(z + a)$ for x ,

$$du + (u + b) dz = m(z + a)^n dz \dots (1),$$

then assume $u = Az^\alpha + Bz^{\alpha+1} + Cz^{\alpha+2} + \dots$;

therefore, differentiating and substituting in (1),

$$\left. \begin{array}{l} \alpha Az^{\alpha-1} + (a+1)Bz^\alpha + (a+z)Cz^{\alpha+1} + \dots \\ + b \quad + Az^\alpha \quad + Bz^{\alpha+1} \quad + \dots \\ -ma^n - mna^{n-1}z - mn \frac{n-1}{z} a^{n-2}z^2 - \dots \end{array} \right\} = 0.$$

Then, in order that this quantity may be equal to nothing for all values of z , a must $= 1$, and the coefficient of each power of z must be separately equal to nothing, therefore we get

$$A = (ma^n - b),$$

$$B = \frac{mna^{n-1} - ma^n + b}{1.2},$$

$$C = \frac{mn(n-1)a^{n-2} - mna^{n-1} + ma^n - b}{1.2.3},$$

$$\dots = \dots$$

therefore we know the coefficients of the assumed series which gives

$$\begin{aligned} y - b &= A(x-a) + B(x-a)^2 + C(x-a)^3 \\ &= (ma^n - b)(x-a) + \frac{mna^{n-1} - ma^n + b}{1.2}(x-a)^2 + \dots \end{aligned}$$

which is the same as that obtained by means of *Taylor's* theorem.

Sometimes the previous transformation of the equation is unnecessary, an arbitrary constant being introduced by the process itself.

Ex. $\frac{d^2y}{dx^2} + cx^ny = 0, \dots (1),$

assume $y = Ax^a + Bx^{a+\delta} + Cx^{a+2\delta} + \dots$

$\therefore \frac{d^2y}{dx^2} = a(a-1)Ax^{a-2} + (a+\delta)(a+\delta-1).Bx^{a+\delta-2}$

$+ (a+2\delta)(a+2\delta-1).Cx^{a+2\delta-2} + \dots$

$cx^ny = c.\{Ax^{a+n} + Bx^{a+\delta+n} + Cx^{a+2\delta+n} + \dots\},$

and the whole sum of these must be indentially nothing. But we see that $a(a-1)Ax^{a-2}$ is the only term involving $x^{(a-2)}$; therefore, unless we were to suppose $n = -2$, which would only be taking a particular case of the given equation, we must have

$$a(a-1)Ax^{a-2} = 0,$$

$$\therefore a = 0, \text{ or } a = 1;$$

then we must have

$$(a+\delta)(a+\delta-1).Bx^{a+\delta-2} + cAx^{a+n} = 0,$$

$$\therefore \delta - 2 = n,$$

$$\text{or } \delta = (n+2),$$

and we shall have for determining the coefficients, the equations

$$(a+\delta)(a+\delta-1).B + cA = 0,$$

$$(a+2\delta)(a+2\delta-1).C + cB = 0,$$

$$\dots \dots \dots = \dots$$

The number of coefficients is greater by unity than the number of equations, therefore one of them will remain arbitrary; also two series will be found, one corresponding to $a = 0$, and the other to $a = 1$. Each of them, then, will contain an arbitrary constant, and therefore will be a particular integral, and their sum will be the complete integral; it is

$$\begin{aligned} \therefore A - \frac{Acx^{n+2}}{(n+1)(n+2)} + \frac{Ac^2x^{2n+4}}{(n+1)(n+2)(2n+3)(2n+4)} \\ - \frac{Ac^3x^{3n+6}}{(n+1)(n+2)(2n+3)(2n+4)(3n+5)(3n+6)} + \dots \\ + A_1x - \frac{cA_1x^{n+3}}{(n+2)(n+3)} + \frac{c^2A_1x^{2n+5}}{(n+2)(n+3)(2n+4)(2n+5)} \\ - \frac{c^3A_1x^{3n+7}}{(n+2)(n+3)(2n+4)(2n+5)(3n+6)(3n+7)} + \dots, \end{aligned}$$

where A, A_1 are the two arbitrary constants.

There is another method of approximation depending on continued fractions, but as the process does not appear likely to be of much general use, we shall merely refer the reader who is curious about it, to *Lacroix's* large work on the Differential and Integral Calculus; where he will find many details on the subject, which our limits prevent us inserting.

PART II.

ON DIFFERENTIAL EQUATIONS

BETWEEN THREE OR MORE VARIABLES.

CHAPTER I.

DEFINITIONS AND GENERAL CHARACTERS.

WE know, by the Differential Calculus, that if z be a function of two independent variables x and y , there will be two differential coefficients of the first order, viz. $\frac{dz}{dx}$, and $\frac{dz}{dy}$, and three of the second, viz. $\frac{d^2z}{dx^2}$, $\frac{d^2z}{dx dy}$, and $\frac{d^2z}{dy^2}$; and, similarly, if z were a function of three independent variables, there would be three differential coefficients of the first order, and so on.

Hence it appears that the equations arising from the differentiation of functions of three or more variables, will differ materially from those which we have hitherto considered; and, therefore, peculiar methods must be adopted for their solution.

They are generally divided into two classes:—

1st. *Total Differential Equations*, which are those involving the *differentials* of *all* the variables contained; and

2nd. *Partial Differential Equations*, or those involving one or more of the partial differential coefficients of the *dependent* variable.

Thus

$$Pdx + Qdy + Rdz + Sdt = 0$$

is a total differential equation between the four variables x, y, z, t , while

$$P \frac{dz}{dx} + Q \frac{dz}{dy} + R \frac{dz}{dt} = 0$$

is a partial differential equation in which z is the dependent and x, y, t the independent variables.

Of these two classes, the latter (which forms the subject of the third chapter of this part) is by far the most important, inasmuch as partial differential equations are frequently met with in the researches of Natural Philosophy, while total differential equations are of comparatively rare occurrence.

The subject of equations between three or more variables is one of great extent, and if treated in detail, might of itself fill a volume; but, fortunately, those parts of it which are of most practical utility in the *present state* of physical science, may be pursued without reference to the more profound and theoretical investigations of analysts; thus becoming available to the general mathematical student.

CHAPTER II.

ON TOTAL DIFFERENTIAL EQUATIONS.

THE general form of a total differential equation of the first order between three variables, is

$$Pdx + Qdy + Rdz = 0;$$

where P, Q and R are functions of x, y and z . An equation of this form is not always integrable, nor does there always exist a factor which will make it so; for, as we shall presently shew, a certain relation must hold between the coefficients P, Q, R, in order that any one of the variables may be considered as a function of the other two, or (which amounts to the same thing) in order that the equation may admit of an integral of the form

$$f(x, y, z) = C.$$

There are, however, several cases of the equation which are easily integrated.

SECT. 1. *On Equations integrable by the separation of the variables and other algebraic artifices.*

When an equation can by any means be put in the form $Xdx + Ydy + Zdz = 0$, where X, Y, Z are functions of x, y, z respectively, the equation may be immediately integrated, giving

$$\int Xdx + \int Ydy + \int Zdz = C.$$

Ex. 1. $dz + \frac{\sqrt{1+z^2}}{y(1+y^2)}dy + n\sqrt{\frac{1+z^2}{1+x^2}}dx = 0,$

dividing by $\sqrt{1+z^2}$, the equation becomes

$$\frac{dz}{\sqrt{1+z^2}} + \frac{dy}{y(1+y^2)} + \frac{ndx}{\sqrt{1+x^2}} = 0,$$

$$\text{or } \frac{dz}{\sqrt{1+z^2}} + \left(\frac{1}{y} - \frac{y}{1+y^2}\right) dy + \frac{ndx}{\sqrt{1+x^2}} = 0;$$

therefore, integrating

$$\log_e(z + \sqrt{1+z^2}) + \log_e y - \log_e \sqrt{1+y^2} + n \log_e(x + \sqrt{1+x^2}) = \log_e C;$$

$$\therefore \frac{(z + \sqrt{1+z^2}) \cdot (x + \sqrt{1+x^2})^n \cdot y}{\sqrt{1+y^2}} = C.$$

Ex. 2. $(x^2 + y^2) dz = (z - a)(x dx + y dy),$

$$\therefore \frac{dz}{z-a} = \frac{x dx + y dy}{x^2 + y^2},$$

$$\therefore \log_e \{C(z-a)\} = \frac{1}{2} \log_e (x^2 + y^2),$$

$$\therefore C^2(z-a)^2 = x^2 + y^2.$$

The variables are frequently separated by a proper substitution.

Ex. 3. $\frac{xdy - xdz + zdx - ydx}{(x-y+z)^2} = 0,$

then assume $x - y + z = t,$

$$\therefore y = x + z - t, \text{ and } dy = dx + dz - dt;$$

whence, by substitution, the equation gives

$$\frac{tdx - xdt}{t^2} = 0,$$

$$\therefore \frac{x}{t} = C, \text{ or } \frac{x}{x-y+z} = C.$$

SECT. 2. *On the criteria of integrability, and the integration of equations which satisfy them.*

When a total differential equation between three variables is proposed, we have two things to determine about it before we attempt its integration; and these are,—first, whether it is the complete differential of a function $f(x, y, z) = C$; and, secondly, if it be not so, whether there exists any factor which will make it so.

To determine the conditions of integrability of the equation $dz = Pdx + Qdy$, where P and Q do not contain z .

If the equation is integrable, or, in other words, if z can be expressed in a function of x and y , then

$$dz = \frac{dz}{dx} dx + \frac{dz}{dy} dy;$$

therefore, comparing with the given equation

$$P = \frac{dz}{dx}; \quad Q = \frac{dz}{dy},$$

$$\therefore \frac{dP}{dy} = \frac{dz}{dx dy} = \frac{dx}{dy dx} = \frac{dQ}{dx};$$

therefore, the condition is

$$\frac{dP}{dy} = \frac{dQ}{dx},$$

which is the same as with equations of two variables.

COR. The method of integration will also be precisely similar to that for equations between two variables, except that instead of having $f(x, y, C) = 0$ for the integral, we shall have $z = f(x, y, C)$.

Ex. 1. $dz = \frac{dx}{\sqrt{1+x^2}} + adx + 2bydy,$

gives $z = C + \log_e (x + \sqrt{1+x^2}) + ax + by^2.$

Ex. 2. $dz = \frac{1}{y} \left\{ dy + \frac{xydx - y^2dy}{\sqrt{x^2 - y^2}} \right\},$

or $dz = \frac{dy}{y} + \frac{xdx - ydy}{\sqrt{x^2 - y^2}};$

$\therefore z = \log_e y + \sqrt{x^2 - y^2} + C.$

To find the conditions that the equation $Pdx + Qdy + Rdz = 0$ may be immediately integrable.

Suppose the equation to be the complete differential of $f(x, y, z) = C$; then

$$P = \frac{d.f(x,y,z)}{dx}; \quad Q = \frac{d.f(x,y,z)}{dy}; \quad R = \frac{d.f(x,y,z)}{dz},$$

$$\therefore \frac{dQ}{dz} = \frac{d.f(x,y,z)}{dydz} = \frac{d.f(x,y,z)}{dzdy} = \frac{dR}{dy};$$

$$\therefore \frac{dQ}{dz} = \frac{dR}{dy};$$

and proceeding in a similar manner with the other differential coefficients, we should find, on the whole, the three equations of condition,

$$\frac{dQ}{dz} = \frac{dR}{dy}; \quad \frac{dP}{dz} = \frac{dR}{dx}; \quad \frac{dP}{dy} = \frac{dQ}{dx}.$$

Given the equation $Pdx + Qdy + Rdz = 0$ to find the condition that it may be a perfect differential, or that there may exist a factor which will make it so.

Suppose μ the factor in question,

$$\therefore \mu Pdx + \mu Qdy + \mu Rdz = 0$$

is to be a complete differential; and on the separate suppositions of x, y, z being constant, we have respectively the equations

$$\mu Qdy + \mu Rdz = 0,$$

$$\mu Pdx + \mu Rdz = 0,$$

$$\mu Pdx + \mu Qdy = 0,$$

which must each of them be perfect differentials; therefore, we must have

$$\frac{d.\mu R}{dy} = \frac{d.\mu Q}{dz}; \quad \frac{d.\mu R}{dx} = \frac{d.\mu P}{dz}; \quad \frac{d.\mu Q}{dx} = \frac{d.\mu P}{dy},$$

or, performing the differentiations,

$$\left. \begin{aligned} \mu \left(\frac{dR}{dy} - \frac{dQ}{dz} \right) + R \frac{d\mu}{dy} - Q \frac{d\mu}{dz} &= 0, \\ \mu \left(\frac{dR}{dx} - \frac{dP}{dz} \right) + R \frac{d\mu}{dx} - P \frac{d\mu}{dz} &= 0, \\ \mu \left(\frac{dQ}{dx} - \frac{dP}{dy} \right) + Q \frac{d\mu}{dx} - P \frac{d\mu}{dy} &= 0. \end{aligned} \right\} (A).$$

Now, multiplying the first of these equations by P , the second by $-Q$, and the third by R , and adding, the sum is divisible by μ , and gives us the equation of condition,

$$P \frac{dR}{dy} - R \frac{dP}{dy} + R \frac{dQ}{dx} - Q \frac{dR}{dx} + Q \frac{dP}{dz} - P \frac{dQ}{dz} = 0,$$

which being independent of μ , shews that whatever μ be, the equation cannot become a perfect differential, unless this relation exist between P, Q, R ; or, in other words, that unless this relation exist, we are not at liberty to suppose one of the variables a function of the other two.

COR. 1. When the condition is fulfilled, the determination of μ depends on any two of the three equations (A).

COR. 2. The equation of condition may be put in the form

$$P \left(\frac{dR}{dy} - \frac{dQ}{dz} \right) + Q \left(\frac{dP}{dz} - \frac{dR}{dx} \right) + R \left(\frac{dQ}{dx} - \frac{dP}{dy} \right) = 0,$$

in which each term will separately equal 0, if the equation is integrable without the introduction of a factor, for each of the quantities in the brackets will equal 0, as has been shewn in page 108.

To integrate the equation $Pdx + Qdy + Rdz = 0$, when it satisfies the criterion of integrability. (first method.)

Consider x and z variable, but y constant, and our equation becomes

$$Pdx + Rdz = 0,$$

and integrating on this supposition, a function of y must be added instead of a constant of integration;

$$\therefore \int Pdx + \int Rdz + Fy = \text{integral} = f(x, y, z) \text{ suppose,}$$

$$\text{or } \phi(x, y, z) + \psi(x, y, z) + Fy, = f(x, y, z) \dots (a).$$

Then, differentiating this with respect to y , and representing the differential coefficient of $\psi(x, y, z)$ with respects to y , by $\psi'y$, ... we have

$$\phi'y + \psi'y + F'y = \frac{df(x, y, z)}{dy},$$

$$\text{or } \phi'y + \psi'y + F'y = Q;$$

$$\therefore Fy = \int dy \{Q - \phi'y - \psi'y\} + C$$

is determined, and its value, substituted in equation (a), will give us the complete integral.

Ex. $\frac{ydx}{(a-z)} + \frac{xdy}{(a-z)} + \frac{xydz}{(a-z)^2} = 0.$

Then, supposing y constant, and integrating, we have

$$\begin{aligned} \frac{xy}{(a-z)} + Fy &= \text{the integral} \dots\dots (a), \\ &= f(x, y, z) \text{ suppose;} \end{aligned}$$

therefore, differentiating with respect to y

$$\begin{aligned} \frac{x}{(a-z)} + F'y &= \frac{df(x, y, z)}{dy} = \frac{x}{a-z}; \\ \therefore F'y &= 0, \\ \therefore Fy &= C, \end{aligned}$$

and substituting this in (a), we have for the integral

$$\frac{xy}{a-z} + C = 0.$$

To integrate the equation $Pdx + Qdy + Rdz = 0$. (second method.)

Suppose $du = Pdx + Qdy + Rdz$, where u is some function of x, y, z .

Then we may assume

$$u = \int Pdx + F(z, y);$$

$$\therefore = \phi(x, y, z) + F(z, y),$$

where ϕ is a function known from the integration, and $F(z, y)$ is a function to be determined.

Then, differentiating with respect to y and z separately, we have

$$\frac{du}{dy} = \phi'y + F'y = Q,$$

$$\frac{du}{dz} = \phi'z + F'z = R,$$

representing the differential coefficients as before; and from these two equations, $F'z$ and $F'y$ being found, $F(z, y)$ may be determined, and therefore

$$u = \phi(x, y, z) + F(z, y)$$

will be known.

Ex.
$$\frac{ydy + xdx + zdz}{\sqrt{y^2 + x^2 + z^2}} + \frac{zdx - xdz}{x^2 + z^2} + zdx = 0.$$

In this example it will be simplest to suppose x and z constant;

$$\begin{aligned}\therefore u &= \int \frac{ydy}{\sqrt{y^2 + x^2 + z^2}} + F(z, x), \\ &= \sqrt{y^2 + x^2 + z^2} + F(z, x),\end{aligned}$$

$$\therefore \frac{du}{dx} = \frac{x}{\sqrt{y^2 + x^2 + z^2}} + F'(x) = \frac{z}{x^2 + z^2} + \frac{x}{\sqrt{y^2 + x^2 + z^2}},$$

$$\therefore F'x = \frac{z}{x^2 + z^2};$$

$$\text{also, } \frac{du}{dz} = \frac{z}{\sqrt{y^2 + x^2 + z^2}} + F'z = \frac{z}{\sqrt{x^2 + y^2 + z^2}} - \frac{x}{x^2 + z^2} + z,$$

$$\therefore F'z = z - \frac{x}{x^2 + z^2}.$$

Now, integrating $F'x$, considering z constant, we have

$$F(x, z) = \tan^{-1} \frac{x}{z} + Z;$$

therefore, differentiating with respect to z ,

$$F'z = -\frac{x}{x^2 + z^2} + \frac{dZ}{dz},$$

$$\therefore \frac{dZ}{dz} = z, \therefore Z = \frac{z^2}{2} + C;$$

$$\text{hence } F(x, z) = \tan^{-1} \frac{x}{z} + \frac{z^2}{2} + C,$$

$$\therefore u = \sqrt{y^2 + x^2 + z^2} + \tan^{-1} \frac{x}{z} + \frac{z^2}{2} + C = 0$$

is the required integral.

If the equation $Pdx + Qdy + Rdz = 0$, be homogeneous, and also satisfies the three equations of condition in page 108, it may be immediately integrated; but if it only satisfies the one equation of condition in page 109, we must find a factor which will make it integrable.

To find the integrating factor; suppose the complete primitive to be $u = f(x, y, z) = C$. Then, since C is arbitrary, this may take the form $f(x, y, z) = 0$, which being homogeneous, may be put in the forms $f\left(\frac{x}{y}, \frac{z}{y}\right) = a$; $\psi\left(\frac{y}{x}, \frac{z}{x}\right) = b$; and, in order that these may hold, the variables must be so connected, that if x become $(1+a)x$ and y become $(1+a)y$, z becomes $(1+a)z$, and therefore u becomes $(1+a)^n u$ where n is the dimensions of u .

Then, by the expansion of a function of three variables,

$$(1+a)^n u = u + \frac{du}{dx} ax + \frac{du}{dy} ay + \frac{du}{dz} az, \\ + \text{terms of } a^2, a^3 \dots$$

and since a is indeterminate, we may equate coefficients of like powers of a .

$$\therefore nu = \frac{du}{dx} x + \frac{du}{dy} y + \frac{du}{dz} z, \\ = \mu Px + \mu Qy + \mu Rz,$$

if μ be the factor which makes the equation a perfect differential.

$$\text{Hence } \frac{du}{nu} = \frac{Pdx + Qdy + Rdz}{Px + Qy + Rz},$$

therefore, the integrating factor is $\frac{1}{Px + Qy + Rz}$.

COR. If $Pdx + Qdy + Rdz = 0$ satisfies the three equations of condition, page 108, it is a perfect differential; and, therefore,

$$nu = Px + Qy + Rz;$$

or if m be the dimensions of P , Q , R , we have for the integral

$$u = \frac{Px + Qy + Rz}{m + 1} = C.$$

Ex. The homogeneous equation

$$(y + z)dx + (x + z)dy + (x + y)dz = 0$$

satisfies the criterion of integrability, and, in this case, $m = 1$; therefore the integral will be

$$\frac{(y + z)x + (x + z)y + (x + y)z}{2} = C,$$

$$\text{or } xy + yz + xz = C.$$

It is evident that this method may fail when $m = -1$, for then the denominator in the above will become $= 0$; hence it is not applicable when the equation has been made a perfect differential, by means of the factor found as above.

SECT. 3. *On Equations which do not satisfy the criterion of integrability.*

As we have already shewn, unless the coefficients of the equation $Pdx + Qdy + Rdz = 0$ have a certain relation to each other, we are not at liberty to assume that any one of the variables is a function of the other two, considered as independent of each other; and, moreover, that no factor can be introduced, which will make such an assumption allowable.

For a long time equations not satisfying this condition were looked upon as paradoxical, and devoid of any geometrical signification; but *Monge* has shewn, in the *Memoires de l'Académie des Sciences de Paris*, 1784, that all differential equations between three variables admit of geometrical interpretation; and that these, in particular, represent

(each) an infinity of curves of double curvature, to which some common property belongs.

To find the geometrical signification of an equation which does not satisfy the criterion.

Suppose $Pdx + Qdy + Rdz = 0$ the equation. Then, since we are not at liberty to suppose z a function of x and y , while x and y are independent of each other, we must assume some relation between the variables, so as to find two equations between them involving an arbitrary function.

Ex. 1. $dz = xy(xdx + ydy)$ does not satisfy the criterion; it may be put in the form

$$dz = d\left(\frac{xy^3}{3}\right) + \left(yx^2 - \frac{y^3}{3}\right)dx,$$

and this equation will be integrable if x and y be so connected that

$$yx^2 - \frac{y^3}{3} = Fx \dots\dots\dots (1)$$

therefore, assuming this to be the case, the given equation gives the integral

$$z = \frac{xy^3}{3} + \int Fx.d x \dots\dots\dots (2).$$

The equations (1) and (2) satisfy the differential equation, and in them the form F is arbitrary, and, therefore, an infinite number of pairs of equations (1) and (2) may be formed, which will satisfy the differential equation, and each pair will belong to a curve of double curvature.

Again, the parts of the expressions which are unconnected with F , do not change when the form F is changed; and, therefore, the curves will have some common property belonging to them.

Ex. 2. $dz = aydx + bdy$ will be integrable, if

$$y = Fx;$$

$$\text{but if } y = Fx, \quad dy = Fx.d x;$$

therefore, substituting and integrating, our two equations are

$$\left. \begin{aligned} y &= Fx \\ x &= a \int Fx \cdot dx + b \cdot Fx \end{aligned} \right\}$$

and these will satisfy the differential equation, whatever form be assigned to F .

Ex. 3. $xdx + y(dx + dy) = 0,$

assume $x + y = u,$

and our equation becomes

$$xdx + ydu = 0,$$

and if we assume $x = Fu$, this gives

$$\{x F'u + y\} du = 0,$$

$$\therefore x F'u + y = 0;$$

therefore, the two equations containing the integral of the given equation are

$$\left. \begin{aligned} x F'(x + y) + y &= 0 \\ z &= F(x + y) \end{aligned} \right\}.$$

We shall easily perceive that the given differential equation will result from the elimination of the arbitrary function; for from the second we get, differentiating,

$$dz = F'(x + y)(dx + dy),$$

and substituting the value of $F'(x + y)$ from the first

$$dz = -\frac{y}{x}(dx + dy),$$

which coincides with the given equation.

CHAPTER III.

ON PARTIAL DIFFERENTIAL EQUATIONS.

A partial differential equation is said to be of the n^{th} order, and m^{th} degree, when at least one of the differential coefficients rises to the n^{th} order, the exponent of the highest power of any of them being m . This chapter, therefore, will naturally divide itself into sections corresponding to the different forms.

SECT. 1. *On the nature and general properties of partial Differential Equations.*

It has already been stated, that a partial differential equation is one involving the partial differential coefficients of one of the variables taken with respect to the others considered as *independent* of each other; hence, to perform their integration, we are at liberty, as in the preceding chapter, to assume either of the independent variables constant, always remembering to add an arbitrary function of it, instead of a constant of integration.

There exists, however, one important difference between partial and total differential equations; namely, that in the complete integral of the former, there will always be one or more arbitrary functions introduced, and, therefore, this integral will represent some one whole class of surfaces, while the latter generally give integrals, each of which belongs to one particular surface.

To shew how the arbitrary functions are to be accounted for.

Suppose we have an equation between x, y and z , in which z is considered a function of the independent variables x and y , and that the equation, besides involving definite functions of x and y , also involves m arbitrary functions of these variables.

Represent the equation by

$$F_1 \{x, y, z, f(P), \phi(Q), \dots\} = 0 \dots (1),$$

where P and Q are known functions, but the forms $f, \phi \dots$ are arbitrary, and may be any whatever; then, differentiating with respect to x , and representing

$$\frac{d.f(P)}{dP} \text{ by } f'(P), \frac{d.\phi(Q)}{dQ} \text{ by } \phi'(Q), \dots$$

we should obtain an equation of the form

$$F_2 \left\{ x, y, z, \frac{dz}{dx}, f'(P) \frac{dP}{dx}, \phi'(Q) \frac{dQ}{dx} \dots \right\} = 0 \dots (2);$$

also, differentiating with respect to y , we should have

$$F_3 \left\{ x, y, z, \frac{dz}{dy}, f'(P) \frac{dP}{dy}, \phi'(Q) \frac{dQ}{dy} \dots \right\} = 0 \dots (3).$$

Hence we have introduced m more arbitrary functions $f'(P), \phi'(Q) \dots$ and have obtained two more equations; similarly, differentiating these two equations again, we should obtain three more equations respectively involving

$$\frac{d^2z}{dx^2}, \frac{d^2z}{dxdy}, \text{ and } \frac{d^2z}{dy^2},$$

and we should have m more arbitrary functions, viz. $f''(P), \phi''(Q) \dots$; and, similarly, proceeding to the n^{th} differentiations, we should have $(n + 1) m$ arbitrary functions, and

$$1 + 2 + 3 + \dots + (n + 1) \text{ equations.}$$

We might, therefore, eliminate all the arbitrary functions obtaining a partial differential equation of the n^{th} order, provided

$$1 + 2 + 3 + \dots + (n + 1) \text{ is not less than } (n + 1) m + 1;$$

that is, if $(3 + n) \frac{n}{2}$ be not less than $(n + 1)m$,

or, if m be not greater than $\frac{(3 + n)n}{2(n + 1)}$.

Hence a partial differential equation of the n^{th} order will have an integral involving $\frac{(3 + n)n}{2(n + 1)}$ arbitrary functions, and there may be more, for it is evident that the elimination might, in some cases, be effected before the number of equations exceeded the whole number of arbitrary functions.

In general it will be found, that the number of arbitrary functions introduced is equal to the exponent of the order of the differential equation; this, perhaps, arises from some symmetry in the form of the primitive, which, while it renders the equation integrable by the known processes, enables us, by differentiation, to eliminate the arbitrary functions before the number of equations is greater than the number of such functions.

In each integration performed on a partial differential equation of the n^{th} order, (as we shall see) one arbitrary function is introduced; and, therefore, on the whole, n such functions will appear; but it has been observed by *Poisson*, that whenever the differential coefficient relative to one of the independent variables is of a higher order than the differential coefficient relative to the other, the number of arbitrary functions in the integral may be reduced to the number expressing the order of the last mentioned differential coefficient.

As an example of this, we shall shew that the complete solution of the equation $\frac{d^2 z}{dy^2} = \frac{dz}{dx}$ (which is not integrable in finite terms, by the ordinary methods) may be so modified, as to involve only one arbitrary function; for, representing by (z) , $\left(\frac{dz}{dy}\right)$, $\left(\frac{d^2 z}{dy^2}\right)$, ... the values of z , $\frac{dz}{dy}$, $\frac{d^2 z}{dy^2}$, ... when

$y = 0$, we have, by *Maclaurin's* theorem,

$$z = (z) + y \left(\frac{dz}{dy} \right) + \frac{y^2}{1.2} \left(\frac{d^2z}{dy^2} \right) + \frac{y^3}{1.2.3} \left(\frac{d^3z}{dy^3} \right) + \dots$$

or, comparing with the given equation,

$$z = fx + y\phi x + \frac{y^2}{1.2} \frac{d.fx}{dx} + \frac{y^3}{1.2.3} \frac{d.^2fx}{dx^2} + \dots$$

which involves two arbitrary functions. But we may assume

$$fx = A + Bx + \frac{Cx^2}{1.2} + \frac{Dx^3}{1.2.3} + \dots$$

$$\phi x = A_1 + B_1x + \frac{C_1x^2}{1.2} + \frac{D_1x^3}{1.2.3} + \dots$$

$$\begin{aligned} \therefore z &= A + A_1y + \frac{By^2}{1.2} + \frac{B_1y^3}{1.2.3} + \frac{Cy^4}{1.2.3.4} + \frac{C_1y^5}{1.2.3.4.5} + \dots \\ &\quad + x \left(B + B_1y + \frac{Cy^2}{1.2} + \frac{C_1y^3}{1.2.3} + \dots \right) \\ &\quad + \frac{x^2}{1.2} \left(C + C_1y + \dots \right) + \dots \end{aligned}$$

which will be found to be of the form

$$z = Fy + x \cdot \frac{d.^2Fy}{dy^2} + \frac{x^2}{1.2} \frac{d.^4Fy}{dy^4} + \dots$$

and, therefore, the complete solution obtained is reduced to one involving the single arbitrary function Fy .

SECT. 2. *On partial Differential Equations of the first order and degree.*

We shall first occupy ourselves with some particular forms easily integrable.

To integrate the equation $\frac{dz}{dx} = P$ when P is a function of x, y, z .

If we consider y constant, we may integrate the equation between the two variables z and x , by the known methods; but we must add a function of y , ϕy , instead of a constant of

integration, ϕ signifying a function absolutely arbitrary, and as general as it is possible to conceive.

$$\text{Ex. } \frac{dz}{dx} = \frac{y^2 + z^2}{y^2 + x^2};$$

$$\therefore \frac{dz}{y^2 + z^2} = \frac{dx}{y^2 + x^2},$$

and, integrating on the supposition that y is constant, we have

$$\tan^{-1} \frac{z}{y} - \tan^{-1} \frac{x}{y} = \phi y;$$

$$\therefore \frac{z - x}{y^2 + xz} = \tan. \phi y = fy;$$

$$\therefore z = \frac{y^2 fy + x}{1 - xfy}.$$

To integrate the equation $\frac{dz}{dy} = P$, the process will be exactly similar; x is now to be considered constant in the integration, and an arbitrary function of x is to be added.

$$\text{Ex. } \frac{dz}{dy} = \frac{y(x+z)}{\sqrt{x^2 + y^2}},$$

$$\therefore \frac{dz}{x+z} = \frac{ydy}{\sqrt{x^2 + y^2}},$$

$$\therefore \log_e (x+z) = \sqrt{x^2 + y^2} + fx.$$

To integrate the general equation of the first order.

$$\frac{dz}{dx} + M \frac{dz}{dy} = N,$$

where M and N are functions of x , y , and z .

Assume $F(x, y, z) = 0$ to be the primitive, and representing the differential coefficient of this with respect to x by F_x , and similarly for the other variables, we have

$$\frac{dz}{dx} = - \frac{\text{differential coefficient with respect to } x}{\text{differential coefficient with respect to } z} = - \frac{F_x}{F_z};$$

$$\text{and, similarly, } \frac{dz}{dy} = - \frac{F_y}{F_z}.$$

Then, substituting these values in the given equation, and multiplying the whole by $F'z$,

$$F'x + MF'y + NF'z = 0, \\ \therefore F'x = -(MF'y + NF'z).$$

But

$$d.F(x, y, z) = F'x.dx + F'y.dy + F'z.dz, \\ = -(MF'y + NF'z) dx + F'y.dy + F'z.dz; \\ = F'y(dy - Mdx) + F'z(dz - Ndx).$$

Now the only condition required is, that this should equal zero, which it evidently will if we are able to assign any relations between the variables x, y, z , which will simultaneously satisfy the two equations

$$dy - Mdx = 0; \quad dz - Ndx = 0,$$

and these two equations will give us a third, viz :

$$dz - \frac{N}{M} dy = 0.$$

In the first place, suppose we are able to integrate any two of these equations, obtaining the integrals $P = a$, $Q = b$, where a and b are constants of integration. Then, by means of the two relations thus established, the values of x, y, z , might all be expressed in terms of the constants a and b ; and one of the variables; and supposing their values substituted in $F(x, y, z)$, it would take the form $F(x, a, b)$. But we know that this quantity is to equal zero, and this cannot be the case *generally*, unless the terms of x are such as to destroy each other, therefore the integral is $F(a, b) = 0$, in which we must put the values of a and b in terms of the variables, whence we have

$$F(P, Q) = 0, \text{ or } Q = \phi P,$$

which is the required primitive.

Secondly, if only one of the three assumed equations can be integrated, we may, generally, from its integral, determine some relation between two of the variables which will enable us to eliminate one of them from one of the remaining equations, thus rendering it integrable; and if the two equations give the

integrals P and Q, the primitive of the given equation will, as in the previous case, be $Q = \phi P$.

COR. It is evident that the integrability of the equations, and therefore the success of the method, depends on M not containing z , or N not containing y , or $\frac{N}{M}$ not containing x .

Ex. 1. Let the equation be

$$x \frac{dz}{dx} + y \frac{dz}{dy} = nz,$$

therefore the two first equations give us

$$dy - \frac{y}{x} dx = 0, \quad dz - \frac{nz}{x} dx = 0,$$

which are both integrable, and give

$$\frac{y}{x} = a, \quad \frac{z}{x^n} = b,$$

therefore the required integral is

$$\frac{z}{x^n} = \phi\left(\frac{y}{x}\right), \text{ or } z = x^n \phi\left(\frac{y}{x}\right).$$

Ex. 2. Let $x \frac{dz}{dx} + z \frac{dz}{dy} + y = 0$;

the three equations become

$$xdy - zdx = 0; \quad xdz + ydx = 0; \quad xdz + ydy = 0,$$

of which only one is integrable, giving

$$z^2 + y^2 = a^2, \text{ suppose,}$$

and this integral gives us $y = \sqrt{a^2 - z^2}$, therefore, substituting this value in the second equation, it becomes

$$xdz + dx \sqrt{a^2 - z^2} = 0,$$

$$\therefore \frac{dz}{\sqrt{a^2 - z^2}} + \frac{dx}{x} = 0;$$

$$\therefore \sin^{-1} \frac{z}{a} + \log_e x = \log_e b, \text{ suppose,}$$

$$\therefore b = x e^{\sin^{-1} \left(\frac{z}{a} \right)} = \phi(y^2 + z^2),$$

or substituting in the first side the value of a in terms of the variables, we have for the complete integral

$$xe^{\sin^{-1} \frac{z}{\sqrt{z^2+y^2}}} = \phi(y^2+z^2).$$

Similarly, we may integrate the equation

$$\frac{dz}{dx} + M \frac{dz}{dy} + N \frac{dz}{dt} = P,$$

where the coefficients are functions of x, y, z, t , of which x, y , and t are considered the independent variables.

Assume $F(x, y, z, t) = 0$ to be the integral, then

$$\frac{dz}{dx} = -\frac{F_x}{F_z}; \quad \frac{dz}{dy} = -\frac{F_y}{F_z}; \quad \frac{dz}{dt} = -\frac{F_t}{F_z};$$

therefore, substituting in the equation, it gives

$$F_x = -(P F_z + M F_y + N F_t).$$

But $dF(x, y, z, t) = F_x dx + F_y dy + F_z dz + F_t dt$; therefore, substituting the value of F_x ,

$$F_z (dz - P dx) + F_y (dy - M dx) + F_t (dt - N dx) = 0.$$

Now that this should equal 0 is the only condition required, and therefore, as in the former case, we may assume each term to equal 0, which gives the three equations $dz - P dx = 0$; $dy - M dx = 0$; $dt - N dx = 0$; and, from these, others may be formed by combining them two and two.

Then, if by any means we can obtain from three of the equations the three integrals $P = a$, $Q = b$, $R = c$, the function $F(x, y, z, t)$ might be expressed in terms of one variable, and the constants a, b, c ; and, therefore, as before, in order that we may have the function equal 0 for all values of this one variable, the terms containing it must of themselves vanish.

Hence we shall have $f(a, b, c) = 0$, or putting the values of a, b, c in terms of the variables, our integral will be $f(P, Q, R) = 0$, or, which comes to the same thing,

$$R = \phi(P, Q).$$

Ex. $\frac{dx}{dx} + \frac{z+t}{x} \frac{dz}{dy} + \frac{z+y}{x} \frac{dt}{dt} = \frac{t+y}{x},$

then the three assumed equations become

$$dz - \frac{t+y}{x} dx = 0; \quad dy - \frac{z+t}{x} dx = 0; \\ dt - \frac{z+y}{x} dx = 0;$$

multiplying them all by x , we obtain

$x dz - x dy - y dx + z dx = 0$; subtracting 2nd from 1st.

$x dy - x dt - t dx + t dx = 0$; subtracting 3rd from 2nd.

$x (dz + dy + dt) - 2(x + y + t) dx = 0$; adding all three.

The three integrals obtained from these are

$$xz - xy = a; \quad xy - xt = b, \quad \frac{x + y + t}{x^2} = C;$$

therefore the integral of the given equation is

$$\frac{x + y + t}{x^2} = \phi \{ (xz - xy), (xy - xt) \},$$

$$\text{or } x + y + t = x^2 \cdot \phi \{ x(z - y), x(y - t) \}.$$

SECT. 3. *On partial Differential Equations above the first degree.*

In the integration of these equations it will be useful to represent the partial differential coefficients $\frac{dz}{dx}$ and $\frac{dz}{dy}$ by p, q , respectively.

This being premised, we know that if z be a function of x and y ,

$$dz = p dx + q dy;$$

which equation, being integrated *by parts*, gives the three following,

$$z = px + \int (q dy - x dp) \dots \dots \dots (\alpha),$$

$$z = qy + \int (p dx - y dq) \dots \dots \dots (\beta),$$

$$z = px + qy - \int (x dp + y dq) \dots \dots (\gamma).$$

We shall frequently be able to facilitate the integration of our equation by substituting the value of p or q , obtained from it, in one of the above expressions.

To integrate an equation of the form

$$\frac{dz}{dx} = F\left(\frac{dx}{dy}\right), \text{ or } p = Fq,$$

substitute Fq for p in the expression (γ) , and we have

$$z = xFq + yq - \int (xF'q + y) dq;$$

and, on the hypothesis that this is integrable, $xF'q + y$ must be a function of q without x and y , which condition gives us

$$xF'q + y = \phi'q \dots \dots \dots (1),$$

$$\therefore z + \int \phi'q \cdot dq = xFq + yq,$$

$$\text{or } z + \phi q = xFq + yq \dots \dots \dots (2).$$

Then if the form ϕq be assigned at pleasure, or determined from the nature of the problem that gave rise to the differential equation, the form $\phi'q$ can be obtained by differentiation, and the integral of the proposed equation will result from the elimination of q between the equations (1) and (2).

$$\text{Ex. } 1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2 = m^2, \text{ or } 1 + p^2 + q^2 = m^2;$$

$$\therefore p = \sqrt{m^2 - 1 - q^2};$$

$$\therefore z = x\sqrt{m^2 - 1 - q^2} + yq - \int \left(y - \frac{xq}{\sqrt{m^2 - 1 - q^2}}\right) dq;$$

therefore we must have

$$y - \frac{xq}{\sqrt{m^2 - 1 - q^2}} = \phi'q,$$

$$\text{and } z + \phi q = x\sqrt{m^2 - 1 - q^2} + yq;$$

therefore, assigning the form of ϕ , and determining that of ϕ' , q might be eliminated.

As a particular case of the equation, suppose ϕq constant $= C$, $\therefore \phi'q = 0$; then, eliminating q from the equations

$$\left. \begin{aligned} z + C &= x\sqrt{m^2 - 1 - q^2} + yq \\ \text{and } y - \frac{xq}{\sqrt{m^2 - 1 - q^2}} &= 0 \end{aligned} \right\}$$

we find $z + C = \sqrt{(m^2 - 1)(x^2 + y^2)}$, which evidently satisfies the equation.

To integrate an equation of the form

$$f\left(\frac{dz}{dx}, x\right) = F\left(\frac{dz}{dy}, y\right).$$

$$\text{Suppose } f\left(\frac{dz}{dx}, x\right) = \theta, \therefore F\left(\frac{dz}{dy}, y\right) = \theta,$$

from which equations we shall obtain

$$\frac{dz}{dx} = \psi(x, \theta), \quad \frac{dz}{dy} = \Psi(y, \theta),$$

$$\therefore dz = \psi(x, \theta) dx + \Psi(y, \theta) dy.$$

Now, suppose u and v to be the integrals of $\psi(x, \theta) dx$, and $\Psi(y, \theta) dy$, taken on the supposition that θ is constant; then, differentiating these again, we have

$$du + dv = \psi(x, \theta) dx + \Psi(y, \theta) dy + \frac{d(u + v)}{d\theta} d\theta,$$

$$\therefore dz = du + dv - \frac{d(u + v)}{d\theta} d\theta,$$

and, in order that this should be an exact differential, we must have

$$\left. \begin{aligned} \frac{d(u + v)}{d\theta} &= \phi'\theta \\ \therefore z &= u + v - \phi\theta \end{aligned} \right\} \dots (a);$$

then, eliminating θ from these equations, after having assigned the form ϕ , and, therefore, found the form ϕ' by differentiation, we shall have the required integral.

$$\text{Ex. } a^2 \frac{dz}{dx} \cdot \frac{dz}{dy} = x^2 y^2,$$

$$\therefore \frac{x^2}{ap} = \frac{aq}{y^2}.$$

Then, assuming each of these quantities to equal θ , we have

$$\frac{dz}{dx} = \frac{x^2}{a\theta}, \text{ and } \frac{dz}{dy} = \frac{y^2\theta}{a};$$

and integrating these quantities, the first with respect to x as the only variable, and the second with respect to y ,

$$u = \frac{x^3}{3a\theta}, \quad v = \frac{y^3\theta}{3a},$$

$$\therefore \frac{d(u+v)}{d\theta} = \frac{y^3}{3a} - \frac{x^3}{3a\theta^2} = \phi'\theta,$$

$$z = \frac{x^3}{3a\theta} + \frac{y^3\theta}{3a} - \phi\theta.$$

Then, if the form of $\phi\theta$ were assigned, that of $\phi'\theta$ would be found by differentiating, and, therefore, θ might be eliminated between the two last equations.

To integrate the general equation $F(x, y, z, p, q) = 0$.

From the given equation, find q in terms of the other quantities involved, and substitute in the equation

$$dz = p dx + q dy.$$

Then the problem reduces itself to the finding what function of x, y, z must be substituted for p to make this latter integrable, which it can only be when

$$\frac{dp}{dy} + \frac{dp}{dz} \frac{dz}{dy} - \frac{dq}{dx} - \frac{dq}{dz} \frac{dz}{dx} = 0,$$

$$\text{or } \frac{dq}{dx} - \frac{dp}{dy} + p \frac{dq}{dz} - q \frac{dp}{dz} = 0;$$

and, since q and p are functions of x, y , and z , by differentiating the given equation, we should have

$$dq = A dp + B dx + C dy + D dz \dots (1);$$

$$\therefore \frac{dq}{dx} = A \frac{dp}{dx} + B; \quad \frac{dq}{dz} = A \frac{dp}{dz} + D;$$

and, therefore, the equation of condition may be put in the form

$$A \frac{dp}{dx} - \frac{dp}{dy} + (pA - q) \frac{dp}{dz} + B + pD = 0 \dots (2).$$

This equation may be treated by the method in page 123, the three equations obtained from it being

$$\left. \begin{aligned} A dp + (B + pD) dx &= 0 \\ A dy + dx &= 0 \\ A dz + (pA - q) dx &= 0 \end{aligned} \right\} \dots (a).$$

The value of q in terms of p, x, y, z , is to be substituted

in these, and their integration will give a value of p in terms of x, y, z , which, substituted in the equation

$$dz = p dx + q dy,$$

will make it integrable, giving the integral belonging to the proposed equation.

Ex. $yq^2 = p^2x^2,$

whence $q = \frac{px}{\sqrt{y}};$

$$\therefore A = \frac{x}{\sqrt{y}}; B = \frac{p}{\sqrt{y}}; C = -\frac{px}{2y^{\frac{3}{2}}};$$

and, since the value of q does not contain z , the three equations (a) are reduced to two, which are

$$\left. \begin{aligned} \frac{xdp}{\sqrt{y}} + \frac{pdx}{\sqrt{y}} &= 0 \\ \frac{xdy}{\sqrt{y}} + dx &= 0 \end{aligned} \right\}.$$

These give us the integrals

$$px = a, \quad \log_e x + 2\sqrt{y} = b,$$

therefore, the value of p , to be substituted in the equation $dz = p dx + q dy$, is

$$p = \frac{1}{x} \phi \{ \log_e x + 2\sqrt{y} \},$$

and the value of $q = \frac{1}{\sqrt{y}} \phi \{ \log_e x + 2\sqrt{y} \}$, therefore the equation becomes

$$\begin{aligned} dz &= \phi \{ \log_e x + 2\sqrt{y} \} \left(\frac{dx}{x} + \frac{dy}{\sqrt{y}} \right) \\ &= \phi \{ \log_e x + 2\sqrt{y} \} \times d. (\log_e x + 2\sqrt{y}); \end{aligned}$$

$$\therefore z = \psi \{ \log_e x + 2\sqrt{y} \},$$

when ψ is a perfectly arbitrary function; and $\sqrt{y}$ may be taken $\pm \sqrt{y}$. (See *Appendix*, No. X).

SECT. 4. *On partial Differential Equations of the second and higher orders.*

There are some general forms of partial differential equations which are integrated without much difficulty.

$$\text{First, } F \left\{ x, y, z, \frac{dz}{dx}, \frac{d^2z}{dx^2}, \dots, \frac{d^n z}{dx^n} \right\} = 0,$$

which may be integrated n times successively, considering y constant, and adding at each integration an arbitrary function of y .

$$\text{Secondly, } F \left\{ x, y, z, \frac{dz}{dy}, \frac{d^2z}{dy^2}, \dots, \frac{d^n z}{dy^n} \right\} = 0,$$

which is to be integrated on the supposition that x is constant, adding n arbitrary functions of x .

$$\text{Thirdly, } F \left\{ x, y, \frac{d^n z}{dy^n}, \frac{d^{n+1}z}{dx dy^n}, \dots, \frac{d^{n+m}z}{dx^m dy^n} \right\} = 0,$$

which is reduced to the first form, with v in the place of z , by putting $v = \frac{d^n z}{dx^n}$.

Having next found v in terms of x and y , we shall have to integrate the equation $v = \frac{d^n z}{dx^n} = f(x, y)$, n times, on the supposition that y is constant, each time adding an arbitrary function of y .

The following are particular cases of the above.

To integrate the equation $\frac{d^2z}{dx dy} = M$, when M is a function of x and y without z .

Put $\frac{dz}{dx} = p$, therefore our equation becomes

$$\frac{dp}{dy} = M,$$

then integrating this, on the supposition that x is constant, we have

$$p = \int M dy + \phi'x,$$

$$\text{or } \frac{dz}{dx} = \int M dy + \phi'x;$$

therefore, integrating again, on the supposition that y is constant,

$$z = \int dx \int M dy + \int \phi' x dx + fy,$$

$$\text{or } z = \int dx \int M dy + \phi x + fy.$$

It is evident, that if it were more convenient, we might first integrate with respect to x , and then with respect to y , obtaining

$$z = \int dy \int M dx + fy + \phi x.$$

Ex. Let $\frac{d^2 z}{dx dy} = ax + by,$

$$\therefore \frac{dz}{dx} = axy + \frac{by^2}{2} + \phi' x,$$

$$\therefore z = \frac{ax^2 y}{2} + \frac{by^2 x}{2} + \phi x + fy,$$

$$= \frac{xy}{2} (ax + by) + \phi x + fy.$$

To integrate the equation $\frac{d^2 z}{dx^2} = P \frac{dz}{dx} + Q$, where P and Q are functions of x and y .

This equation is linear with respect to $\frac{dz}{dx}$, and gives $dp = (Pp + Q) dx$, the integrating factor for which is $e^{-\int P dx}$.

Integrating, we have

$$p = \frac{dz}{dx} = e^{\int P dx} \{ \int e^{-\int P dx} Q dx + \phi y \},$$

where an arbitrary function of y is added instead of a constant, because y has been considered constant in the integration.

Now this equation must be again integrated with respect to x , and another arbitrary function of y must be added, when we shall have a value of z , which is the required primitive.

Ex. $xy \frac{d^2 z}{dx^2} = (n-1) y \frac{dz}{dx} + a,$

$$\text{or } \frac{d^2 z}{dx^2} = \frac{(n-1)}{x} \frac{dz}{dx} + \frac{a}{xy}.$$

Comparing this with the above, we have

$$P = \frac{n-1}{x}; \quad Q = \frac{a}{xy};$$

and the first integration gives

$$\frac{dz}{dx} = -\frac{a}{(n-1)y} + x^{n-1} \phi y,$$

$$\therefore z = -\frac{ax}{(n-1)y} + \frac{x^n \phi y}{n} + \psi y.$$

To integrate the equation $\frac{d^2z}{dx dy} = M \frac{dz}{dx} + N$, where M and N are functions of x and y , we have only to put $\frac{dz}{dx} = p$, and the equation becomes

$$\frac{dp}{dy} = Mp + N,$$

which is linear; therefore, integrating first with respect to y , and then with respect to x , and adding the proper arbitrary functions, we shall have the required integral.

Ex. Let $\frac{d^2z}{dx dy} = \frac{bdz}{ydx} + \frac{a}{x}$,

then integrating first with respect to y ,

$$\frac{dz}{dx} = -\frac{ay}{(b-1)x} + y^b \phi' x,$$

$$\therefore z = -\frac{ay \log x}{(b-1)} + y^b \phi x + \psi y.$$

To integrate the general equation,

$$\frac{d^2z}{dx^2} + S \frac{d^2z}{dx dy} + T \frac{d^2z}{dy^2} + V = 0,$$

where S, T, V, are functions of x, y, z, p , and q .

Let $\frac{d^2z}{dx^2} = r$, $\frac{d^2z}{dx dy} = s$, $\frac{d^2z}{dy^2} = t$;

then $dp = rdx + sdy \dots\dots\dots (a)$,

$dq = sdx + tdy \dots\dots\dots (b)$;

also $dz = pdx + qdy$, therefore, differentiating,

$$d^2z = rdx^2 + 2sdx dy + tdy^2.$$

Then, if from equations (a) and (b) we get values of r and t , and substitute in the given equation, which is

$$r + Ss + Tt + V = 0,$$

it becomes

$$\frac{dp - sdy}{dx} + Ss + T \cdot \frac{dq - sdx}{dy} + V = 0,$$

or

$$dpdy + Tdqdx + Vdxdy - s(dy^2 - Sdxdy + Tdx^2) = 0 \dots (1),$$

which equation is satisfied by the suppositions

$$\left. \begin{aligned} dpdy + Tdqdx + Vdxdy &= 0 \\ dy^2 - Sdxdy + Tdx^2 &= 0 \end{aligned} \right\} \dots (A).$$

If, in the latter of these, we make $dy = kdx$, we shall find two values of k , which are the roots of the equation

$$k^2 - Sk + T = 0;$$

therefore, if these roots be k, k' , we have

$$dy - kdx = 0; \quad dy - k'dx = 0;$$

and the first of equations (A), by the substitution of kdx for dy , gives

$$kdp + Tdy + Vkdx = 0.$$

Then suppose

$$m.(kdp + Tdy + Vkdx) + n(dy - kdx) = 0,$$

and let m and n be so taken that this is a complete differential, giving the integral $F = a$; and evidently this integral will satisfy the given equation. Again, if by any other pair of values of m and n , we can obtain another integral $G = \beta$, this will also satisfy the proposed equation.

Moreover, taking any constant a ,

$$dF + adG \text{ is an exact differential,}$$

and the integral $F + aG = c$ will satisfy the proposed equation a and c , being two constants; but, also, if a and c are variable the above integral will satisfy the equation, provided that we equate to zero the result of the differentiation with respect to a and c , that is, provided that

$$Gda = dc;$$

whence we conclude that G and c are to be functions of a ; and, therefore, a and c are functions of G ;

$$\begin{aligned}\therefore F &= c - aG, \\ &= \phi G,\end{aligned}$$

will satisfy the proposed equation ϕ , being an arbitrary function.

This will then be one of the complete first integrals, since F and G contain p and q , and proceeding in a similar manner with k' instead of k , we should find the other complete first integral $F_1 = \psi G_1$, therefore the two first integrals are found, and from these the other integrals are to be obtained.

Ex. To integrate the equation

$$\frac{d^2z}{dx^2} - a^2 \frac{d^2z}{dy^2} = 0.$$

Comparing it with the equation

$$r + Ss + Tt + V = 0,$$

$$\text{we have } S = 0, \quad T = -a^2, \quad V = 0,$$

and the equations (A) become, in this case,

$$dpdy - a^2 dq dx = 0,$$

$$dy^2 - a^2 dx^2 = 0;$$

from the last we have for $dy = kdx$

$$dy = \pm a dx;$$

therefore $\pm a$ are the values of k as above; therefore, taking the first of them, the equation (B) becomes

$$m(adp - a^2 dq) + n(dy - a dx) = 0,$$

and the system of values $m = 0, n = 1$, gives the integral

$$y - ax = G;$$

and the system $m = 1, n = 0$, gives

$$p - aq = F,$$

therefore our first integral is

$$(p - aq) = \phi'(y - ax) \dots (1).$$

Similarly, proceeding with the second value of k , or $-a$, we get

$$p + aq = f'(y + ax) \dots (2).$$

These are the two first integrals, and we have now to integrate them in order to find the primitive.

By addition and subtraction

$$p = \frac{1}{2} \{ \phi'(y - ax) + f'(y + ax) \},$$

$$q = \frac{1}{2a} \{ f'(y + ax) - \phi'(y - ax) \};$$

then, substituting these values in the equation,

$$dz = p dx + q dy,$$

$$dz = \frac{1}{2} \phi'(y - ax) dx - \frac{1}{2a} \phi'(y - ax) dy,$$

$$+ \frac{1}{2a} f'(y + ax) dy + \frac{1}{2} f'(y + ax) dx,$$

$$\therefore z = -\frac{1}{2a} \phi(y - ax) + \frac{1}{2a} f(y + ax)$$

$$\text{or } z = F(y + ax) + \psi(y - ax).$$

The equation to a vibrating chord is usually written

$$\frac{d^2 y}{dt^2} = b^2 \frac{d^2 y}{dx^2},$$

which is of the same form as the above, and therefore gives the integral

$$y = F(x + bt) + \psi(x - bt).$$

SECT. 5. *On the different kinds of solutions of partial Differential Equations.*

In the integration of total differential equations there is a constant to be added, and the same differential equation generally gives rise to the same primitive, or one differing from it only in the particular value of the arbitrary constant. This, however, is not the case with partial differential equations, the integrals of which have a much more general signification than any algebraic equation from which such differential equations could be obtained.

For instance, the partial differential equation

$$z = x \frac{dz}{dx} + y \frac{dz}{dy} \dots \dots (\alpha),$$

being integrated by the method in page 122, gives the

integral $z = x\phi\left(\frac{y}{x}\right)$, which includes an infinity of algebraic equations, among which are

$$z = ax + by, \text{ and } z = ay + \frac{bx^2}{y};$$

whereas, the expression (a) probably originated from some particular one of these, any of which may be considered the complete primitive.

Suppose the primitive, from which a given equation has arisen, to be

$$U = F(x, y, z, a, b) = 0,$$

where a and b are constants; then the differential equation would arise from the elimination of a and b between the equations

$$\frac{dU}{dx} + \frac{dU}{dz} \frac{dz}{dx} = 0,$$

$$\frac{dU}{dy} + \frac{dU}{dz} \frac{dz}{dy} = 0, \text{ and } U = 0.$$

If, now, a and b be supposed variable instead of constant, our differentiations would give us the same results, provided we had

$$\frac{dU}{da} = 0, \text{ and } \frac{dU}{db} = 0. \dots (\beta);$$

and, hence, if we assume these two equations, and from them determine $a = f(x, y, z) = P$, and $b = \phi(x, y, z) = Q$, and substitute P and Q for a and b in U , we shall have a function

$$F(x, y, z, P, Q) = 0,$$

which will give rise to the same differential equation. This function is called the *singular primitive*.

Again, we may shew that the primitive of the given differential equation may have a still more general form than that just investigated; for we may assume $b = \phi a$, (where ϕ denotes any function whatever), which value substituted, would give

$$U = F(x, y, z, a, \phi a) = 0 \dots (\gamma);$$

and if, as before, this equation be differentiated, and the differential coefficient $\frac{dU}{da} + \frac{dU}{d(\phi a)} \frac{d(\phi a)}{da}$ be equated to zero, we should obtain a value of a in terms of the variables, such that when substituted in the primitive (γ), the derived equation would be the same as if a had remained constant. The function thus obtained would be $F(x, y, z, P, \phi P)$, which is called the *general primitive*; P denoting the value of a , in terms of x, y, z , found from the last condition.

Supposing the complete primitive $U = F(x, y, z, a, b) = 0$ to belong to a curve surface; it is required to explain the geometrical significations of the general and singular primitives.

First, we know that the *general primitive* results from the elimination of a between the equations

$$F(x, y, z, a, \phi a) = 0, \text{ and } \frac{d}{da} F(x, y, z, a, \phi a) = 0;$$

or, which is the same thing, for the elimination of a between the equations $U = 0$, and $\frac{dU}{da} = 0$, because, in this case, a is considered the only independent parameter.

But supposing $U = 0$ to be the equation to a surface, if in this equation we suppose a to become $a + \delta a$, the equation

$$U + \frac{dU}{da} \delta a + \dots = 0$$

would belong to another surface of the same nature as the former one, but differing from it in the magnitude of the parameter a , and therefore, generally, intersecting it; and these two equations, considered as simultaneous, would determine the intersection of the two surfaces, in which case they would reduce themselves to

$$U = 0; \quad \frac{dU}{da} \delta a + \frac{d^2 U}{da^2} \frac{(\delta a)^2}{1.2} + \dots = 0;$$

or in the limit, when the values of a are supposed consecutive, or δa indefinitely small,

$$U = 0, \text{ and } \frac{dU}{da} = 0,$$

would be the equations to the intersection of the two consecutive surfaces corresponding to the particular value of a .

This intersection is called the *characteristic* of the surfaces to which the equation $U = 0$ belongs.

Now the *general primitive* results from the elimination of a between these two equations, and it is evident, therefore, that it must be the equation to the locus of all the *lines* of intersection corresponding to all values of a .

Again, the general primitive gives the same values of $\frac{dz}{dx}$ and $\frac{dz}{dy}$ that $U = 0$ does, and, therefore, the surface represented by it will touch all the surfaces included in $U = 0$, in the lines of their mutual intersections; hence it is called the envelope.

Secondly, to explain the singular primitive; suppose that in the equation

$$U = 0 \dots \dots \dots (1)$$

a and b are independent of each other, and that a becomes $a + \delta a$, while b remains the same; then the equation to the corresponding surface is

$$U + \frac{dU}{da} \delta a + \dots = 0 \dots \dots (2);$$

and if b is supposed to become $b + \delta b$, while a remains the same, we should have a third surface represented by the equation

$$U + \frac{dU}{db} \delta b + \dots = 0 \dots \dots (3).$$

Now if we suppose δa and δb indefinitely small, the three equations

$$U = 0; \quad \frac{dU}{da} = 0; \quad \frac{dU}{db} = 0;$$

would determine the point of intersection of the three surfaces in question; but, if instead of determining x, y, z from these, we eliminate a and b between them, the result must be the equation to the locus of all the *points* in which the surfaces included in $U = 0$ intersect each other, three and three together: hence the surface represented by the singular primitive, is the locus of all such points, for, by our definition, the singular primitive is the result of this elimination.

As, in the first case, the singular primitive will give the same values of $\frac{dz}{dx}$ and $\frac{dz}{dy}$ as the complete primitive; and, therefore; the surface represented by it will touch all the surfaces included in $U = 0$ at their points of mutual intersection.

APPENDIX.

EXAMPLES ON DIFFERENTIAL EQUATIONS.

I. Equations in which the variables are easily separable,
(Chap. II. Sect. 1).

$$(1). \quad (x^2 + 2xy + y^2) \frac{dy}{dx} = b^2.$$

Put $x + y = u$, and the equation becomes

$$u^2(du - dx) = b^2 dx,$$

$$\therefore bx + c = \tan^{-1} \frac{x+y}{b}, \text{ or } y = b \tan (bx + c) - x.$$

$$(2). \quad (e^x + y) dx - \left(\frac{y}{e^x} + 1 \right) dy = 0.$$

Multiplying by $\frac{e^x}{e^x + y}$ we get $dy = e^x dx$,

$$\therefore y = e^x + c.$$

$$(3). \quad y(1 - \cos x) dx - \sin x dy = 0.$$

Dividing by $y \sin x$, each term becomes integrable; and if $\log c$ be put for the constant, we get

$$\frac{\tan \frac{x}{2}}{\sin x} = cy,$$

$$\therefore y = \frac{1}{2c} \sec^2 \frac{x}{2}.$$

$$(4). \quad dy - (x + y)^n dx = b^n dx.$$

Putting $x + y = z$, and $1 + a^n = b^n$, we get

$$\frac{dz}{b^n + z^n} = dx,$$

in which the variables are separated; in the case in which $n = 2$, we get

$$x + y = (1 + a^2)^{\frac{1}{2}} \cdot \tan.x \sqrt{1 + a^2}.$$

(5). To integrate the equation

$$\frac{dx}{\sqrt{(A+Bx+Cx^2+Ex^3+Fx^4)}} + \frac{dy}{\sqrt{(A+By+Cy^2+Ey^3+FY^4)}} = 0.$$

Consider x and y as functions of t , and make

$$\frac{dx}{dt} = \sqrt{(A+Bx+Cx^2+Ex^3+Fx^4)} = \sqrt{X},$$

$$\text{and } \frac{dy}{dt} = \sqrt{(A+By+Cy^2+Ey^3+FY^4)} = \sqrt{Y}.$$

Also, assume $x + y = p$, and $x - y = q$;

$$\therefore \text{ since } \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = X + Y,$$

$$\text{and } \frac{d^2x}{dt^2} + \frac{d^2y}{dt^2} = \frac{dX}{2dx} + \frac{dY}{2dy},$$

we have

$$\frac{d^2p}{dt^2} = B + Cp + \frac{3}{2}E(p^2 + q^2) + \frac{1}{2}Fp(p^2 + 3q^2) \dots (1),$$

$$\frac{dpdq}{dt^2} = X - Y = Bq + Cpq + \frac{1}{2}Eq(3p^2 + q^2) + \frac{1}{2}Fpq(p^2 + q^2) \dots \dots \dots (2).$$

Then multiply equation (1) by q , and subtract equation (2) from it, and multiplying the remainder by $\frac{2dp}{q^3}$, and integrating, we get

$$\frac{dp^2}{q^2 dt^2} = C + Ep + Fp^2,$$

$$\frac{dp}{dt} = \sqrt{X} + \sqrt{Y} = q \sqrt{(c + Ep + Fp^2)},$$

$$\text{or } \sqrt{X} + \sqrt{Y} = (x - y) \sqrt{\{c + E(x + y) + F(x + y)^2\}}$$

is the integral of the given equation.

Similarly, the integral of the equation

$$\frac{dx}{\sqrt{X}} - \frac{dy}{\sqrt{Y}} = 0$$

$$\text{is } \sqrt{X} - \sqrt{Y} = (x - y) \sqrt{\{c + E(x + y) + F(x + y)^2\}}.$$

It is shewn in works on the Integral Calculus, that $\frac{dx}{\sqrt{X}}$ is always reducible to the form $\frac{k d\theta}{\sqrt{1 - c^2 \sin^2 \theta}}$, where k and c are constants, and c is always less than 1.

By this transformation the equation becomes

$$\frac{d\theta}{\sqrt{1 - c^2 \sin^2 \theta}} \pm \frac{d\phi}{\sqrt{1 - c^2 \sin^2 \phi}} = 0;$$

and its integral found as above, is

$$\sqrt{1 - c^2 \sin^2 \theta} \pm \sqrt{1 - c^2 \sin^2 \phi} = c_1 \sin(\theta \pm \phi).$$

II. Equations which satisfy the criterion of integrability.
(Chap. II. Sect. 2.)

$$(1). \quad (a^2 y + x^3) dx + (b^3 + a^2 x) dy = 0,$$

$$\therefore u = a^2 y x + \frac{x^4}{4} + Y,$$

$$= b^3 y + a^2 x y + X;$$

therefore, comparing, the integral is

$$a^2 x y + b^3 y + \frac{x^4}{4} = c.$$

$$(2). \quad x^3 dy + (3x^2 y - a^3) dx = 0,$$

$$\text{gives } x^3 y - a^3 x = c.$$

$$(3). \quad \frac{dy}{\sqrt{1 - x^2}} + \frac{xy - a}{(1 - x^2)^{\frac{3}{2}}} dx = 0,$$

$$\text{gives } y - ax = c \sqrt{1 - x^2}.$$

$$(4). \quad \frac{dy}{y} = \frac{x dy - y dx}{y \sqrt{x^2 + y^2}}$$

$$\text{gives log. } \left(\frac{x + \sqrt{x^2 + y^2}}{c} \right) = 0, \text{ or } x + \sqrt{x^2 + y^2} = c.$$

$$(5). \quad (\sin y + y \cos x) dx + (\sin x + x \cos y) dy = 0,$$

gives $x \sin y + y \sin x = c$.

III. Linear equations of the first order, and others reducible to that form. (Chap. II. Sect. 5.)

(1). $dy + ydx = ax^2dx$, is linear, the integrating factor being e^x ; $\therefore ye^x = a \int x^2 e^x dx$;

$$\therefore y = ax^3 - 3ax^2 + 6ax - 6a + ace^{-x}.$$

(2). $dy - \frac{x+1}{x} ydx = x^2dx$ is linear, the integrating factor being $\frac{1}{xe^x}$,

$$\therefore y = cxe^x - x^2 - x.$$

$$(3). \quad dx + 2x dy = x^2 y^2 dy.$$

$$\text{But } \frac{1}{x} = v, \therefore dv - 2vdy = -y^2 dy$$

is a linear equation, whence

$$\frac{4-x}{x} = 2(y^2 + y) + 4cxe^{2y}.$$

$$(4). \quad ydy - \frac{by^2}{a} dx = -\frac{m}{a} xdx$$

becomes linear by putting $y^2 = u$,

$$\therefore du - \frac{2b}{a} udx = -\frac{2m}{a} xdx,$$

$$\therefore u = y^2 = \frac{mx}{b} + \frac{ma}{2b^2} + ce^{\frac{2bx}{a}}.$$

$$(5). \quad (x^2 + y^2) dx + (a + x) ydy = 0$$

becomes linear if we put $y^2 = u$, and dividing by $a + x$, we get

$$du + \frac{2udx}{a+x} = -\frac{x^2dx}{a+x};$$

$$\therefore y^2(a+x)^2 = c - \frac{ax^3}{3} - \frac{x^4}{4}.$$

(6). $dx - xydy = x^2y^3dy$; when $\frac{1}{x}$ is put $= v$, gives a linear equation in v ;

$$\text{whence } \frac{1}{x} = \frac{y^2 + 1}{2} + ce^{y^2}.$$

(7). $dy - x^2dy + xydx = adx$ is linear in y , and when divided by $(1 - x^2)$, gives the integral

$$y = ax + c\sqrt{1 - x^2}.$$

The following equations are all to be done by similar means :

$$(8). \quad dy + ydx = xy^3dx; \text{ put } \frac{1}{y^2} = u.$$

$$(9). \quad \frac{ay^2dx}{\sqrt{x}} - 2(x + ay\sqrt{x})dy = 0.$$

$$(10). \quad (1 - x^2)dy + (xy - a)dx = 0.$$

$$(11). \quad (x^2 + y^2)dx + x^2ydy = 0.$$

$$(12). \quad dy - (1 + xy)dx = 0.$$

$$(13). \quad \frac{mdx}{\sqrt{x}} - (y^2 + \sqrt{x})dy = 0.$$

$$(14). \quad (y^n - x)dy + ydx = 0.$$

$$(15). \quad \frac{dy}{y} - \frac{dx}{x} = \frac{x^m dx}{ay}.$$

$$(16). \quad dy - bydx = a \sin xdx.$$

IV. Homogeneous Equations, and others which may be transformed so as to become homogeneous. (Chap. II. Sect. 4.)

(1). $\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 - \frac{y}{x}$ is homogeneous.

Putting $\frac{y}{x} = z$, and separating the variables, we have

$$\frac{dx}{x} = \frac{dz}{z(z^2 - 2)} = -\frac{1}{2} \frac{dz}{z} + \frac{1}{2} \left\{ \frac{dz}{z + \sqrt{2}} + \frac{dz}{z - \sqrt{2}} \right\};$$

$$\therefore x = c \frac{\left\{ \left(\frac{y}{x}\right)^2 - 2 \right\}^{\frac{1}{2}}}{\left(\frac{y}{x}\right)^{\frac{1}{2}}},$$

$$\text{or } y = \frac{c^2 x \sqrt{2}}{\sqrt{c^4 - x^4}}.$$

(2). $2\sqrt{x} dy = \sqrt{y} dx + \sqrt{y} dy.$

Putting $\frac{x}{y} = z$, and separating the variables

$$\frac{dy}{y} + \frac{dz}{(\sqrt{z} - 1)^2} = 0;$$

$$\therefore \log_e y = \log_e \frac{c}{(\sqrt{z} - 1)^2} + \frac{2}{\sqrt{z} - 1},$$

$$\text{whence } \sqrt{y} = \sqrt{x} - \sqrt{c} e^{\frac{\sqrt{y}}{\sqrt{x} - \sqrt{y}}}.$$

(3). $\frac{dx}{x} - \frac{dy}{y} = n \left(\frac{dx}{y} - \frac{dy}{x} \right).$

Put $\frac{x}{y} = z$, and we get the equation

$$\frac{1 - nz}{z^2 - 1} dz = n \frac{dy}{y},$$

$$\therefore \frac{\sqrt{z-1}}{\sqrt{z+1} (z^2-1)^{\frac{n}{2}}} = c y^n,$$

$$\text{or } (x+y)(x^2-y^2)^{\frac{n-1}{2}} = \frac{1}{c}.$$

(4). $xdx - 2ydx + ydy = 0.$

Put $\frac{y}{x} = z$, and we get the equation

$$\frac{dx}{x} + \frac{zdz}{(z-1)^2} = 0,$$

$$\text{whence } c(y-x) = e^{\frac{x}{y-x}}.$$

$$(5). \quad ydx - xdy = dx\sqrt{x^2 + y^2}.$$

Put $\frac{y}{x} = z$, and we get the equation

$$\frac{dx}{x} + \frac{dz}{\sqrt{1+z^2}} = 0;$$

$$\therefore y + \sqrt{x^2 + y^2} = c,$$

which is reducible to $x^2 + 2cy - c^2 = 0$.

$$(6). \quad (xy + y^2) dx + x^2 dy = 0.$$

Put $\frac{y}{x} = z$, and we find

$$\frac{dx}{x} + \frac{dz}{2z + z^2} = 0;$$

$$\therefore \frac{y}{2x + y} = \frac{c^2}{x^2}, \text{ or } y = \frac{2c^2x}{x^2 - c^2}.$$

$$(7). \quad (6x - 9y + 3) dy + (4x - 6y - 10) dx = 0.$$

This equation belongs to the case in page 22, where $na = mb$; proceeding as there directed, we get

$$6dx - \frac{z+3}{z-6} dz = 0,$$

where $z = 6x - 9y$; hence the integral is

$$c(6x - 9y - 6) e^y = 0.$$

$$(8). \quad (7x + 5y + 3) dy + (28x + 20y - 7) dx = 0$$

is done by the same method as the last.

The following are examples of *Riccati's* equation, pages 22 and 23.

$$(9). \quad dy + \left(y^2 - \frac{a^2x}{x^2}\right) dx = 0.$$

Here the exponent of $x = -4$, which is of the form

$$\frac{-4n}{2n-1} \text{ when } n = 1.$$

Then assume $y = \frac{1}{x} - \frac{z}{x^2}$, and $x = \frac{1}{t}$, and transforming the equation, and integrating, we shall obtain

$$e^z \cdot \left\{ \frac{x^2 y - x + a}{x^2 y - x - a} \right\} = c.$$

$$(10). \quad dy + \left(y^2 - \frac{a^2 x}{x^{\frac{5}{2}}} \right) dx = 0.$$

The exponent of $x = -\frac{5}{2}$ comes under the form $\frac{-4n}{2n+1}$, therefore assume $x = t^{-2}$, $y = -3a^2 z^{-1}$;

$$\therefore dz + \left(z^2 - \frac{9a^2}{t^4} \right) dt = 0,$$

which equation is of the same form as the previous example; and integrating and substituting, we have

$$e^{-6ax^{\frac{3}{2}}} \cdot \left\{ \frac{y(1 + 3ax^{\frac{1}{2}}) + 3a^2 x^{-\frac{1}{2}}}{y(1 - 3ax^{\frac{1}{2}}) + 3a^2 x^{-\frac{1}{2}}} \right\} = c.$$

V. Equations of degrees superior to the first, and questions relating to singular solutions. (Chap. III. and IV.)

$$(1). \quad \frac{dy}{dx} - x = n \sqrt{1 + \left(\frac{dy}{dx} \right)^2}.$$

This equation is of the form $x = P$, page 46; therefore, differentiating,

$$dx = dp - \frac{npdp}{\sqrt{1+p^2}} \dots \dots \dots (a);$$

$$\therefore dy = p dx = p dp - \frac{np^2 dp}{\sqrt{1+p^2}}.$$

Integrating this last, we should have to eliminate p between the equations (a) and

$$c + y = \frac{p^2}{2} - \frac{np}{2} \sqrt{1+p^2} + \frac{n}{2} \log. (p + \sqrt{1+p^2}),$$

which would give the complete solution of the equation.

$$(2). \quad \left(\frac{dy}{dx}\right)^2 - 2a \left(\frac{dy}{dx}\right) + y^2 = 0$$

$$\text{gives } \frac{dy}{dx} = a \pm \sqrt{a^2 - y^2},$$

and hence we have the two integrals

$$x + \frac{a}{y} = c \pm \left(\frac{\sqrt{a^2 - y^2}}{y} + \sin^{-1} \frac{y}{a} \right);$$

which, combined, becomes

$$\left(x + \frac{a}{y}\right)^2 = c^2 - \left(\frac{\sqrt{a^2 - y^2}}{y} + \sin^{-1} \frac{y}{a} \right)^2.$$

$$(3). \quad \left(\frac{dy}{dx}\right)^3 - (xy + x^2 + y^2) \left(\frac{dy}{dx}\right)^2 \\ + (x^2y + y^3x + x^2y^2) \frac{dy}{dx} - x^2y^3 = 0.$$

It is evident that the values of $\frac{dy}{dx}$, obtained from this equation, will be xy , x^2 , and y^2 ; therefore, integrating with each of these values, we find the three integrals

$$y = e^{\frac{x^2}{2}} + c, \quad y = \frac{x^3}{3} + c, \quad y = -\frac{1}{x} + c,$$

therefore the general integral is

$$(y - e^{\frac{x^2}{2}} - c)(y - \frac{x^3}{3} - c)(y + \frac{1}{x} - c) = 0.$$

$$(4). \quad y \left(\frac{dy}{dx}\right)^2 + 2x \frac{dy}{dx} = y; \text{ gives}$$

$$\frac{dy}{dx} = \frac{-x \pm \sqrt{x^2 + y^2}}{y} \text{ or } \frac{ydy + xdx}{\sqrt{x^2 + y^2}} = \pm dx;$$

$$\therefore \sqrt{x^2 + y^2} = c + x, \text{ or } \sqrt{x^2 + y^2} = c - x,$$

which integrals are to be combined.

$$(5). \quad 1 + \left(\frac{dy}{dx}\right)^2 = \frac{1}{x}; \text{ and putting } \frac{dy}{dx} = p,$$

we have $x = \frac{1}{1+p^2},$

$$y = \int p dx = px - \int x dp;$$

$$\therefore y = \frac{p}{1+p^2} - \tan^{-1} p + c.$$

From these two values for x and y , p being eliminated, we get

$$y = \sqrt{x-x^2} - \tan^{-1} \sqrt{\frac{1-x}{x}} + c.$$

$$(6). \quad x \frac{dy}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}, \text{ or } px = \sqrt{1+p^2}.$$

This gives $x = \frac{\sqrt{1+p^2}}{p},$

and $dy = p dx = -\frac{dp}{p\sqrt{1+p^2}},$

$$\therefore y = \log_e c + \log_e \left(\frac{\sqrt{1+p^2} + 1}{p} \right);$$

$$\therefore e^y = c (x + \sqrt{x^2 - 1}).$$

$$(7). \quad \left(\frac{dy}{dx}\right)^4 + y \left(\frac{dy}{dx}\right)^3 - 1 = 0, \text{ gives}$$

$$y = \frac{1-p^4}{p^3}, \therefore dx = \frac{dy}{p} = -\frac{3dp}{p^5} - \frac{dp}{p};$$

$$\therefore x = \frac{3}{4p^4} - \log_e p + c;$$

and from these values of x and y , p is to be eliminated.

The following equations come under what is called *Clairaut's* form, (page 44).

$$(8). \quad y = x \frac{dy}{dx} + \frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2$$

gives, by differentiation,

$$x + 1 - 2p = 0$$

and substituting the value of p , obtained from this equation, we find the *singular solution*

$$y = \left(\frac{x+1}{2}\right)^2.$$

To prove that this is a singular solution, resolve the given equation with respect to p , and we get

$$p = \frac{x+1}{2} + \sqrt{\frac{(x+1)^2}{4} - y},$$

and if $\left(\frac{x+1}{2}\right)^2 - z$ be put for y , this gives

$$p_1 = \frac{x+1}{2} + z^{\frac{1}{2}},$$

in which the exponent of z is less than unity; and, therefore, the solution obtained is a singular solution.

The general solution is $y = cx + c - c^2$.

$$(9). \quad y \frac{dy}{dx} = (x+a) \left(\frac{dy}{dx}\right)^2 + a$$

may be put in the form

$$y = px + a \frac{p^2 + 1}{p};$$

therefore, to find a singular solution, we have

$$x + a - \frac{a}{p^2} = 0,$$

$$\text{whence } y = 2\sqrt{ax + a^2}.$$

It may be easily shewn, as in the last example, that this is a singular solution.

The complete solution is $y = cx + a \frac{c^2 + 1}{c}$.

$$(10). \quad y + a \left(\frac{ds}{dx}\right)^2 = x \frac{dy}{dx}$$

is equivalent to $y = px - a(p^2 + 1)$;

therefore, differentiating $p = \frac{x}{2a}$, and the singular solution is

$$y = \frac{x^2 - 4a^2}{4a}.$$

The general solution is $y = cx - a(c^2 + 1)$.

$$(11). \quad y = x \frac{dy}{dx} + \sqrt{b^2 - a^2 \left(\frac{dy}{dx}\right)^2}$$

gives for singular solution

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Similarly the following examples may be solved:

$$(12). \quad y = x \frac{dy}{dx} + \frac{a \left(\frac{dy}{dx}\right)^2}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}.$$

$$(13). \quad (x - y) \frac{dy}{dx} = y + (a - x) \left(\frac{dy}{dx}\right)^2,$$

being divided by $1 + \frac{dy}{dx}$, gives $y = px - \frac{ap^2}{1 + p}$.

$$(14). \quad y = x \frac{dy}{dx} + m \left(\frac{dy}{dx}\right)^{\frac{3}{2}}.$$

VI. Particular forms of equations of superior orders,
(Chap. V. Sect. 1).

$$(1). \quad (1 + x) \frac{d^2y}{dx^2} + a \frac{dy}{dx} = 0, \text{ or } (1 + x) \frac{dp}{dx} + ap = 0.$$

$$\text{Integrating } p = \frac{dy}{dx} = \frac{c}{(1 + x)^a};$$

$$\begin{aligned} \therefore y &= \frac{c}{(1 - a)(1 + x)^{a-1}} + c_1, \\ &= \frac{A}{(1 + x)^{a-1}} + c_1. \end{aligned}$$

$$(2). \quad \frac{dy}{dx} \cdot \frac{d^2y}{dx^2} + (2x + a) \cdot \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + 2(2x + a) = 0$$

is equivalent to

$$\left\{ \frac{dy}{dx} + 2x + a \right\} \left(\frac{d^2y}{dx^2} + 2 \right) = 0;$$

and here the second factor is the differential coefficient of the first ;

$$\therefore \frac{dy}{dx} + 2x + a = C,$$

$$\therefore y + x^2 + ax = Cx + C'.$$

$$(3). \quad \frac{dy}{dx} - (x + a) \frac{d^2y}{dx^2} - x \left(\frac{dy}{dx} \right)^2 = 0.$$

Put $p = \frac{dy}{dx}$, and $\frac{1}{p} = u$, and we get the equation

$$du + \frac{u dx}{a + x} = \frac{x dx}{a + x},$$

$$\text{whence } \frac{1}{u} = p = \frac{2a + 2x}{C^2 + x^2};$$

$$\therefore y = \frac{2a}{c} \tan^{-1} \frac{x}{c} + \log_e c_1 (x^2 + c^2).$$

$$(4). \quad \left\{ 1 + \left(\frac{dy}{dx} \right)^2 \right\}^a + 2a^{\frac{1}{2}} \sqrt{a-x} \cdot \frac{d^2y}{dx^2} = 0.$$

$$\text{This gives } \frac{dx}{2a^{\frac{1}{2}} \sqrt{a-x}} + \frac{dp}{(1+p^2)^{\frac{1}{2}}} = 0,$$

$$\text{whence } \frac{a^{\frac{1}{2}} p}{\sqrt{1+p^2}} = \sqrt{a-x} + c \dots (a).$$

Again, $y = px - \int x dp$,

$$= \frac{ap}{1+p^2} - a \tan^{-1} p - \frac{2a^{\frac{1}{2}} c}{\sqrt{1+p^2}} + c_1,$$

and p is to be eliminated by means of equation (a).

$$(5). \quad nx \frac{d^2y}{dx^2} + \sqrt{1 + \left(\frac{dy}{dx} \right)^2} = 0,$$

integrated between the variables p and x , gives

$$c^n x = p + \sqrt{1+p^2},$$

whence we may find p , and we have

$$y = \int p dx, \text{ or } y = c_1 + \frac{nc^n}{2(n+1)} x^{\frac{n+1}{n}} - \frac{n}{2c^n(n-1)} x^{\frac{n-1}{n}},$$

for the complete integral.

$$(6). \quad \frac{d^2 y}{dx^2} - A \left(\frac{dy}{dx} \right)^2 + B \left(\frac{dy}{dx} \right)^3 = 0.$$

$$\text{This give us } dx = \frac{dp}{p^2(A-Bp)},$$

$$\therefore dy = p dx = \frac{dp}{Ap - Bp^2},$$

and integrating this, we get

$$p(c + e^{Ay}) = ae^{Ay};$$

which, being integrated, gives

$$ax = -\frac{c}{A} e^{-Ay} + y + c_1.$$

$$(7). \quad y^2 + 2 \left(\frac{dy}{dx} \right)^2 - y \frac{d^2 y}{dx^2} = 0.$$

Put $y = e^{ux}$, and we have the equation

$$1 + u^2 - \frac{du}{dx} = 0,$$

whence $u = \tan(x + c)$, and we have

$$y = \frac{c_1}{\cos(x + c)} \text{ for the integral.}$$

$$(8). \quad \frac{d^2 y}{ds^2} - \frac{dy}{dx} \cdot \frac{d^2 x}{ds^2} = \frac{1}{1+x^2}.$$

Transforming the equation so as to have x for the independent variable, as in page 58, Ex. 1, we get the equation

$$\frac{dp}{1+p^2} = \frac{dx}{1+x^2};$$

$$\therefore \tan^{-1} p = \tan^{-1} x + \tan^{-1} c,$$

$$\text{or } p = \frac{x+c}{1-cx};$$

$$\therefore cy = -x - \frac{c^2+1}{c} \cdot \log_e \left(\frac{1-cx}{c_1} \right).$$

VII. Linear equations of orders superior to the first.

It has been found necessary to introduce so many examples of these equations in Chap. V., that few will be required here.

$$(1). \quad \frac{d^2y}{dx^2} - k^2y = \cos mx.$$

Proceeding by the general method, we have to assume

$$y = ae^{kx} + be^{-kx},$$

where a and b are functions of x to be determined.

$$\text{The integral is } y = C_1e^{kx} + C_2e^{-kx} + \frac{\cos mx}{m^2 - k^2}.$$

$$(2). \quad \frac{d^2y}{dx^2} - 4 \frac{dy}{dx} + 13y = 0;$$

on assuming $y = e^{hx}$ gives for determining h the equation

$$h^2 - 4h + 13 = 0, \therefore h = 2 \pm 3\sqrt{-1},$$

$$\therefore y = e^{2x}(C_1 \cos 3x + C_2 \sin 3x).$$

$$(3). \quad \frac{d^2y}{dx^2} - 4 \frac{dy}{dx} + 4y = Ax + B$$

gives two values of h each $= 2$,

$$\text{therefore assume } y = (a + bx)e^{2x},$$

and, proceeding by the variation of parameters, we get

$$y = (C_1 + C_2x)e^{2x} + \frac{Ax}{4} + \frac{A+B}{4}.$$

$$(4). \quad \frac{d^2y}{dx^2} + y + A \cos mx + C \cos nx = 0$$

$$\begin{aligned} \text{gives } y = & C_1 \cos x + C_2 \sin x + \frac{A \cos mx}{m^2 - 1} \\ & + \frac{B \cos nx}{n^2 - 1} - \cos x \left(\frac{A}{m^2 - 1} + \frac{B}{n^2 - 1} \right). \end{aligned}$$

$$(5). \quad \frac{d^2y}{dx^2} - 4 \frac{dy}{dx} + 4y = x^2$$

$$\text{gives } y = e^{2x}(C_1 + C_2x) + \frac{x^2}{4} + \frac{x}{2} + \frac{3}{8}.$$

$$(6). \quad \frac{d^2y}{dx^2} + A \frac{dy}{dx} + By = n(n-1)x^{n-2} + nAx^{n-1} + Bx^n.$$

Suppose α and β roots of the equation

$$h^2 + Ah + B = 0; \text{ then we find}$$

$$y = e^{\alpha x} \left\{ \frac{C_1 - e^{-\alpha x} (\beta x^n - nx^{n-1})}{\alpha - \beta} \right. \\ \left. + e^{\beta x} \frac{C_2 - e^{-\beta x} (\alpha x^n - nx^{n-1})}{\beta - \alpha} \right\},$$

$$\text{or } y = ce^{\alpha x} + c_1 e^{\beta x} + x^n.$$

$$(7). \quad \frac{d^2y}{dx^2} + k^2y = e^x \cos kx.$$

Assume $y = a \cos kx + b \sin kx$, and, proceeding by the variation of parameters, we have

$$y = C_1 \cos kx + C_2 \sin kx + \frac{\sin kx}{k} \int e^x \cos^2 kx \cdot dx \\ - \frac{\cos kx}{k} \int e^x \cos kx \cdot \sin kx \cdot dx,$$

$$\text{or } y = C_1 \cos kx + C_2 \sin kx + \frac{e^x}{1 + 4k^2} \{ 2k \sin kx + \cos kx \}.$$

$$(8). \quad \frac{d^2y}{dx^2} + m \frac{dy}{dx} + n^2y = A \sin nx.$$

The equation for determining h is $h^2 + mh + n^2 = 0$; therefore, if α, β be the roots of this equation,

$$y = C_1 e^{\alpha x} + C_2 e^{\beta x} + A \cdot \frac{e^{\alpha x} \int e^{-\alpha x} \sin nx \cdot dx - e^{\beta x} \int e^{-\beta x} \sin nx \cdot dx}{\alpha - \beta} \\ = C_1 e^{\alpha x} + C_2 e^{\beta x} \\ + \frac{A}{(\alpha^2 + n^2)(\beta^2 + n^2)} \{ (\alpha\beta - n^2) \sin nx + n(\alpha + \beta) \cos nx \} \\ = C_1 e^{\alpha x} + C_2 e^{\beta x} - \frac{A}{mn} \cos nx.$$

$$(9). \quad \frac{d^2y}{dx^2} - \frac{2}{x} \frac{dy}{dx} + \frac{2}{x^2} y = \frac{4}{x^2}$$

comes under the case $Q \pm -\frac{P}{x}$ (page 83).

Assuming $y = xx$, the transformed equation is

$$\frac{d^2x}{dx^2} = \frac{4}{x^3},$$

$$\therefore x = \frac{2}{x} + C_1x + C_2 \text{ and } y = 2 + C_1x^2 + C_2x.$$

$$(10). \quad \frac{d^2y}{dx^2} + \frac{2x}{1+x^2} \frac{dy}{dx} + \frac{y}{(1+x^2)^2} = 0.$$

Transforming the equation so that $\theta = \tan^{-1}x$ shall be the independent variable, we get

$$\frac{d^2y}{d\theta^2} + y = 0,$$

$$\therefore y = C_1 \cos \theta + C_2 \sin \theta,$$

$$= \frac{C_1 + C_2x}{\sqrt{1+x^2}}.$$

VIII. Simultaneous equations.

$$(1). \quad \begin{cases} (5x + 2y) dt + dx = 0 \\ (4x + 3y) dt + dy = 0. \end{cases}$$

Here we have a particular case of the equations (1) and (2), page 92; the equations for determining the values of h and k are

$$h = 5 + 4k; \quad hk = 2 + 3k;$$

whence, representing the two values of k by k, k' , and the two values of h by h, h' , we find

$$k \text{ or } k' = \frac{1}{2}, \text{ or } -1; \quad h \text{ or } h' = 7 \text{ or } 1;$$

$$\therefore x = \frac{C_1 e^{-t} + 2C_2 e^{-7t}}{3}, \quad y = \frac{2C_1 e^{-7t} - 2C_2 e^{-t}}{3};$$

and eliminating t , we should get a relation between x and y .

$$(2). \quad \begin{cases} \left(4x + \frac{y}{4}\right) dt + dx = 0 \\ (3y - x) dt + dy = 0. \end{cases}$$

These belong to the case in which the quadratic equation for h and k has equal roots ;

$$\text{we obtain } x = Ce^{-\frac{7}{2}t} - \frac{y}{2} \dots (a);$$

and substituting this value in the second of the given equations, we get the linear equation

$$dy + \frac{7}{2} y dt = Ce^{-\frac{7}{2}t} dt;$$

$$\therefore e^{\frac{7}{2}t} y = Ct + C_1 \dots (\beta),$$

and t is to be eliminated between the equations (a) and (β).

$$(3). \quad \begin{cases} (7x - y) dt + dx = 0 \\ (2x + 5y) dt + dy = 0. \end{cases}$$

We shall find the values of k and h to be impossible ;

$$k \text{ or } k' = -\frac{1}{2} \pm \frac{1}{2} \sqrt{-1}; \quad h \text{ or } h' = 6 \pm \sqrt{-1};$$

$$\therefore x = \frac{e^{-6t}}{2} \cdot \{ (C'' + C') \cos t + (C' - C'') \sin t \}$$

$$y = e^{-6t} \cdot \{ C' \cos t - C'' \sin t \},$$

whence t is to be eliminated.

$$(4). \quad \begin{cases} (44z + 49y) dx + 4dz + 9dy = xdx \\ (34z + 38y) dx + 3dz + 7dy = e^x dx. \end{cases}$$

These equations are of the form solved at page 94 ; and, eliminating successively dy and dz , we get the equations

$$\left. \begin{aligned} (2x + y) dx + dz &= (7x - 9e^x) dx \\ (4x + 5y) dx + dy &= (4e^x - 3x) dx \end{aligned} \right\},$$

and our equation for determining the assumed quantity (β), is

$$\frac{2\beta + 4}{\beta + 5} = \beta, \text{ which gives } \beta = 1 \text{ or } -4.$$

Hence we find

$$y = 4C_1 e^{-6x} + C_2 e^{-x} + \frac{24}{7} e^x - \frac{17}{3} x + \frac{55}{9},$$

$$z = C_1 e^{-6x} - C_2 e^{-x} - \frac{29}{7} e^x + \frac{19}{3} x - \frac{56}{9}.$$

$$(8). \quad \begin{cases} dz + (5z + y) dx = e^x dx \\ dy + (3y - z) dx = e^{2x} dx. \end{cases}$$

Here the equation for determining β , is $\beta = \frac{1+3\beta}{5-\beta}$, which gives two equal values of β , each equal to 1;

$$\therefore z + y = Ce^{-4x} + \frac{e^x}{5} + \frac{e^{2x}}{6} \dots\dots (a).$$

Then taking the value of z from this equation, substituting it in the second of the given equations, and integrating, we find

$$y = \frac{C' + Cx}{e^{4x}} + \frac{e^x}{25} + \frac{7e^{2x}}{36} \dots\dots (\beta).$$

We not unfrequently meet with simultaneous equations, each of which is integrable separately.

$$(6). \quad \begin{cases} \frac{d^2x}{dt^2} = mx \\ \frac{d^2y}{dt^2} = m(y + k). \end{cases}$$

Integrating, we have

$$\left(\frac{dx}{dt}\right)^2 = m(x^2 - C_1^2); \quad \left(\frac{dy}{dt}\right)^2 = m\{(y + k)^2 - C_2^2\};$$

and since $\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$, we get (by combining these equations

and integrating) a relation between x and y .

$$(7). \quad \begin{cases} \frac{d^2x}{dt^2} + m^2x = m^2at \\ \frac{d^2y}{dt^2} + m^2y = 0. \end{cases}$$

These equations are both linear, and being solved, will give two relations between the three quantities x, y, t .

IX. Equations between three variables of the first order and degree.

First. Total differential equations.

$$(1). \quad dx + dy + dz + (x + y + z) dz = 0$$

$$\text{gives } \frac{dx + dy + dz}{x + y + z} + dz = 0;$$

$$\therefore (x + y + z) e^z = c.$$

$$(2). \quad \frac{by - ax}{z^2} dz + \frac{adx - bdy}{z} = 0$$

may be put in the form

$$\frac{az dx - ax dz}{z^2} + \frac{by dz - bz dy}{z^2} = 0;$$

$$\therefore \frac{ax - by}{z} = c.$$

$$(3). \quad (ay - bz) dx + (cz - ax) dy + (bx - cy) dz = 0$$

$$\text{gives } \frac{ay - bz}{cz - ax} = C.$$

$$(4). \quad (x^2 + xy + y^2) dz + (y^2 + yz + z^2) dx + (z^2 + xz + x^2) dy = 0.$$

When z is considered constant, we have the equation

$$\frac{dx}{x^2 + xz + z^2} + \frac{dy}{y^2 + yz + z^2} = 0,$$

$$\text{which gives } \frac{2}{z\sqrt{3}} \left\{ \tan^{-1} \frac{x\sqrt{3}}{2z+x} + \tan^{-1} \frac{y\sqrt{3}}{2z+y} \right\} = Z_1,$$

$$\therefore \frac{xz + yz + xy}{2z^2 + xz + yz - xy} = \tan Z_1 = Z;$$

$$\therefore \frac{xz + yz + xy}{2z^2 + xz + yz - xy} = \frac{C}{z - C}.$$

Secondly. Partial differential equations.

$$(5). \quad x \frac{dz}{dx} = az.$$

Integrating, on the supposition that y is constant,

$$\log_e z = \log_e x^a + \log_e \phi y, \therefore z = x^a \phi y.$$

$$(6). \quad \frac{dz}{dy} = \frac{x^2}{x^2 + y^2} \text{ gives } z = x \tan^{-1} \frac{y}{x} + \phi x.$$

$$(7). \quad \frac{dz}{dx} = nx - z \text{ gives } z = n(x - 1) + e^{-x} \phi y.$$

$$(8). \quad \frac{dz}{dx} = \frac{y}{x + z}, \therefore e^{\frac{z}{y}} (x + y + z) = \phi y.$$

The following are integrable by the method given in page 120.

$$(9). \quad x \frac{dz}{dx} + y \frac{dz}{dy} = n \sqrt{x^2 + y^2}.$$

The equations to integrate are

$$xdy - ydx = 0; \quad xdz - ndx\sqrt{x^2 + y^2} = 0;$$

therefore we may eliminate y from the second equation, by means of the integral $y = ax$ of the first, and we shall find for the integral of the proposed equation

$$z - n\sqrt{x^2 + y^2} = \phi \left(\frac{y}{x} \right).$$

$$(10). \quad y^2 \frac{dz}{dy} - xy \frac{dz}{dx} = \frac{axz}{y}, \therefore \log_e z + \frac{ax}{3y^2} = \phi(x, y).$$

$$(11). \quad y \frac{dz}{dx} + x \frac{dz}{dy} = z, \therefore z = (x + y) \phi (x^2 - y^2).$$

$$(12). \quad (a - x) \frac{dz}{dx} + (b - y) \frac{dz}{dy} = c - z$$

$$\text{gives } \frac{y - b}{z - c} = \phi \left(\frac{x - a}{z - c} \right).$$

$$(13). \quad \frac{dz}{dx} - \frac{dz}{dy} = \frac{z}{x + y} \text{ gives } z = e^{\frac{z}{x+y}} \cdot \phi(x + y).$$

$$(14). \quad x \frac{dz}{dx} + y \frac{dz}{dy} = \frac{z^{n+1} - c^{n+1}}{z^n}$$

gives the integral $\log. \frac{z^{n+1} - c^{n+1}}{z^{n+1}} = \phi\left(\frac{y}{x}\right)$, or

$$z^{n+1} = c^{n+1} + x^{n+1} \psi\left(\frac{y}{x}\right).$$

$$(15). \quad z - x \frac{dz}{dx} - y \frac{dz}{dy} = m(x + y + z)$$

gives $x^{m-1}(x + y + z) = e^{\psi\left(\frac{y}{x}\right)} = \psi\left(\frac{y}{x}\right)$.

X. Partial differential equations not of the first degree.

The example given at page 128 does not *fully* illustrate the method of solution ; in that particular case the equations (a) have enabled us to obtain p and q in explicit functions of x and y , and hence, after the proper substitutions, the actual integration of the equation

$$dz = p dx + q dy$$

has been effected. But, generally, this will not be the case, as we shall find it impossible to solve the equation (2), page 127, without the introduction of some arbitrary function involving p , which function is to be eliminated by differentiation after having assigned its form. The following examples will make this better understood.

We shall represent $\frac{dz}{dx}$ by p and $\frac{dz}{dy}$ by q throughout.

$$(1). \quad q = xp + p^2.$$

Here the three equations (a) become

$$\left. \begin{aligned} (x + 2p) dp + p dx &= 0 \\ (x + 2p) dy + dx &= 0 \\ (x + 2p) dz - p^2 dx &= 0 \end{aligned} \right\} \dots (a).$$

Dividing the first by the second we get $\frac{dp}{dy} = p$, which gives the integral

$$y - \log_e p = a;$$

$$\text{also, } dy = \frac{dp}{p}.$$

Hence the equation $dz - p dx - q dy = 0$

$$\text{becomes } dz - p dx - (x + p) dp = 0,$$

$$\text{or } dz - p dp - d.xp = 0;$$

$$\therefore z - \frac{p^2}{2} - px = b = \phi a,$$

$$\text{or } z - \frac{p^2}{2} - px = \phi (y - \log_e p) \dots (\beta).$$

From this p may be eliminated when the arbitrary function is assigned, for differentiating with the respect to p , we have

$$\frac{1}{p} \phi' (y - \log_e p) - x - p = 0 \dots (\gamma).$$

and if the form ϕ be assigned, that of ϕ' will be known, and the integral of the proposed equation will result from the elimination of p between the equations (β) and (γ) .

$$(2). \quad p^2 + q^2 = 1.$$

The first of equations (a), page 127, gives, in this case,

$$\frac{p dp}{\sqrt{1-p^2}} = 0, \therefore \sqrt{1-p^2} = \text{constant},$$

$$\text{or } p = a, \text{ and } q = \sqrt{1-a^2}.$$

$$\text{The equation } dz - p dx - q dy = 0$$

$$\text{gives } dz - a dx - dy \sqrt{1-a^2} = 0;$$

$$\therefore z - ax - y \sqrt{1-a^2} = b = \phi a,$$

$$\text{and } \frac{ay}{\sqrt{1-a^2}} - x = \phi' a;$$

therefore, assigning the form ϕ , a may be eliminated between the two last equations.

(3). The equation $1 + p^2 + q^2 = m^2$ (treated in page 125 by another method) may be solved by the means employed in the last example; it gives

$$\left. \begin{aligned} z - ax - y\sqrt{m^2 - a^2 - 1} &= \phi a \\ \frac{ay}{\sqrt{m^2 - a^2 - 1}} - x &= \phi' a \end{aligned} \right\}.$$

(4). $q = xy^2z^3p^2.$

The first and third of equations (a) become

$$2xz \frac{dp}{p} + (z + 3px) dx = 0,$$

$$2dz - p dx = 0;$$

then in the second term of the first of these put $\frac{2dz}{dx}$ for p , and dividing by $2xz$,

$$\frac{dp}{p} + \frac{dx}{2x} + \frac{3dz}{z} = 0;$$

therefore, integrating, we have $px^{\frac{1}{2}}z^3 = a$,

$$\text{and, therefore, } q = \frac{a^2y^2}{z^3}.$$

Substituting these values in the equation $dx - p dx - q dy = 0$, and integrating on the supposition that a is constant,

$$\frac{z^4}{4} - 2a\sqrt{x} - \frac{a^2y^3}{3} = b = \phi a,$$

$$\text{and } 2\sqrt{x} + \frac{2ay^3}{3} + \phi'a = 0.$$

From these equations a is to be eliminated, after having assigned the form ϕ .

XI. Partial differential equations not of the first order.
(Part II. Chap. III. Sect. 4).

(1). $\frac{d^2z}{dxdy} = xy$, or $\frac{dq}{dx} = xy$;

$$\therefore q = \frac{x^2 y}{2} + \phi' y = \frac{dz}{dy};$$

$$\therefore z = \frac{x^2 y^2}{4} + \phi y + f x.$$

$$(2). \quad \frac{d^2 z}{dx dy} = \pm \sqrt{x + y + z};$$

$$\therefore \frac{dz}{dy} = \pm \frac{2}{3} (x + y + z)^{\frac{3}{2}} + \phi'(y, z),$$

$$\text{and } z = \pm \frac{2.2}{3.5} (x + y + z)^{\frac{5}{2}} + \phi(y, z) + f(x, z).$$

$$(3). \quad a \frac{d^2 z}{dx^2} = xy; \therefore 6ax = x^3 y + x\phi y + \psi y.$$

$$(4). \quad x \frac{d^2 z}{dx^2} = (n-1) \frac{dz}{dx}, \text{ or } x \frac{dp}{dx} = (n-1)p;$$

$$\therefore \frac{dz}{dx} = x^{n-1} + \phi y,$$

$$\text{and } z = \frac{x^n}{n} + x\phi y + \psi y.$$

$$(5). \quad \frac{d^2 z}{dx^2} = \frac{ax}{\sqrt{x^2 + y^2}};$$

$$\therefore \frac{dz}{dx} = a\sqrt{x^2 + y^2} + \phi y,$$

$$z = \frac{1}{2} ax\sqrt{x^2 + y^2} + \frac{1}{2} ay^2 \log(x + \sqrt{x^2 + y^2}) + x\phi y + \psi y.$$

The following examples belong to the general form

$$r + Ss + Tt + V = 0,$$

the integration of which has been explained in pages 131—133.

(6). Let $r + As + Bt + C = 0$; the quantities A, B, C , being constant.

The equation for determining the values of k is $k^2 - Ak + B = 0$; and, therefore, these values will be constant, and may be represented by α and β .

We then have the equation

$$m(adp + Bdq + Cadx) + n(dy - adx) = 0;$$

$$\therefore m = 0, n = 1 \text{ gives } y - ax = F,$$

$$m = 1, n = 0 \text{ gives } ap + Bq + Cax = G;$$

therefore we have for one of the first integrals

$$ap + Bq + Cax = \phi'(y - ax),$$

and, similarly, for the other value of k we have

$$\beta p + Bq + C\beta x = \psi'(y - \beta x).$$

We may now find the complete primitive, either by integrating one of these first integrals by the method in page 120, or from the last two equations we may find p and q , substitute in the equation

$$dz = pdx + qdy;$$

and integrate.

By either proceeding we shall find

$$z = \phi(y - ax) + \psi(y - \beta x).$$

$$(7). \quad r + 3s + 2t = ax$$

gives two values of k , viz. 1 and 2; and, corresponding to these, we find the two integrals

$$p + q - \frac{ax^2}{2} = \phi'(y - 2x),$$

$$p + 2q - \frac{ax^2}{2} = \psi'(y - x);$$

and substituting the values of p and q , obtained from these in the equation $dz = pdx + qdy$, and integrating, we get

$$\begin{aligned} z &= \frac{ax^3}{6} - \phi(y - 2x) + \psi(y - x), \\ &= \frac{ax^3}{6} + f(y - 2x) + \psi(y - x). \end{aligned}$$

$$(8). \quad r - a^2t - xy = 0$$

gives $k = \pm a$, and we have, with the first value of k ,

$$m(adp - a^2dy - axy dx) + n(dy - adx) = 0,$$

$$m = 0, n = 1 \text{ gives us } y - ax = F;$$

also, since $dy = kdx$, we get

$$\begin{aligned} \int xy dx &= \frac{yx^2}{2} - \frac{1}{2} \int x^2 dy, \\ &= \frac{yx^2}{2} - \frac{ax^3}{6}; \end{aligned}$$

$$\therefore p - aq - \frac{yx^2}{2} + \frac{ax^3}{6} = \phi'(y - ax);$$

and, similarly, with the second value of k ,

$$p + aq - \frac{yx^2}{2} - \frac{ax^3}{6} = \psi'(y + ax),$$

whence we find the complete primitive

$$z = \frac{x^3 y}{6} + \frac{1}{2a} \left\{ \psi(y + ax) - \phi(y - ax) \right\}.$$

When we get but one value of k , and can only find one first integral, we must obtain the primitive by direct integration.

$$(9). \quad r + \frac{2y}{x} s + \frac{y^2}{x^2} t = 0.$$

The equation for k is $\left(k - \frac{y}{x}\right)^2 = 0$; and hence there is only one value of k , viz: $\frac{y}{x}$.

This value gives the equation

$$m \left(\frac{y}{x} dp + \frac{y^2}{x^2} dq \right) + n \left(dy - \frac{y}{x} dx \right) = 0,$$

if $m = 0$, and $n = 1$, we get $\frac{y}{x} = F$,

and proceeding with the values $m = 1$, $n = 0$, we have

$$p + \frac{y}{x} q = \phi \left(\frac{y}{x} \right)$$

for a first integral.

Applying the general method, page 120, to this equation, we have to integrate

$$dy - \frac{y}{x} dx = 0; \text{ and } dx - \phi \left(\frac{y}{x} \right) dx = 0;$$

the first gives $\frac{y}{x} = a$, which being substituted in the second, we have, by integration,

$$z - x\phi a = \beta;$$

$$\therefore z = x\phi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right).$$

$$(10). \quad r - \frac{2p}{q}s + \frac{p^2}{q^2}t = 0$$

gives but one value of k , which is $-\frac{p}{q}$, and we have the equation

$$m. \left\{ -\frac{p}{q} dp + \frac{p^2}{q^2} dq \right\} + n \left(dy + \frac{p}{q} dx \right) = 0;$$

therefore, for $m = 0$ and $n = 1$, we get $qdy + pdx = 0$, or $dz = 0$; $\therefore z = F$.

Again, the system $n = 0$, $m = 1$, gives

$$qdp - pdq = 0 \quad \dots \dots \dots (a),$$

$$\therefore \frac{p}{q} = G = \phi x$$

is one first integral of the given equation.

To find the other, multiply the equation (a) by $\frac{x}{q^2}$, and add $\frac{pdx + qdy}{q} = 0$ to the product;

$$\therefore \frac{x(qdp - pdq)}{q^2} + \frac{pdx}{q} + dy = 0.$$

This may be integrated, on the supposition that z is constant, giving

$$\frac{xp}{q} + y = \psi z,$$

which is the required first integral, and eliminating $\frac{p}{q}$ between the two, we have

$$x.\phi z + y = \psi z.$$

(11). There is a very simple method of integrating the equation $t = a^2r$, (treated in page 133) due to *D'Alembert*.

We have $dp = rdx + sdy$; $dq = sdx + tdy = sdx + a^2r dy$.

Then multiplying the first by a , we have, by addition and subtraction,

$$adp + dq = (ar + s)(dx + ady),$$

$$adp - dq = (ar - s)(dx - ady),$$

and since the first sides of these are perfect differentials, the second sides must be so also;

$$\therefore ap + q = f'(x + ay); \quad ap - q = F'(x - ay),$$

whence, as before, integrating, we have

$$z = \phi(x + ay) + \psi(x - ay).$$

XII. Problems which give rise to differential equations.

(1). To find a curve (convex to the axis) in which the radius of curvature is always equal to the normal.

Since the curve is to be convex to the axis, the expression for the radius of curvature must be taken positive; hence we get the equation

$$y\sqrt{1+p^2} = \frac{(1+p^2)^{\frac{3}{2}}}{\frac{dp}{dx}},$$

and omitting common factors, and substituting dy for pdx , this becomes

$$\frac{dy}{y} = \frac{pdp}{1+p^2};$$

$$\therefore y = a\sqrt{1+p^2};$$

and solving the equation with respect to p , integrating again, and putting b for the second arbitrary constant,

$$x = b + a \log_e (y + \sqrt{y^2 - a^2});$$

$$\therefore y = \frac{1}{2} \cdot \left\{ e^{\frac{x-b}{a}} + a^2 e^{-\frac{x-b}{a}} \right\}$$

is the equation to the required curve.

(2). To find a curve in which the distance from a given point to any point in the curve is always m times the length of the tangent.

Consider the curve referred to the *given point* as pole, and we have the equation

$$\begin{aligned}\frac{rdr}{dp} &= m \sqrt{r^2 + r^4 \left(\frac{d\theta}{dr}\right)^2}; \\ \therefore \frac{dr}{dp} &= m \frac{ds}{dr}, \text{ but } \frac{dr}{dp} = \frac{ds}{rd\theta}, \text{ generally;} \\ \therefore \frac{dr}{r} &= m d\theta; \\ \therefore r &= ce^{m\theta}\end{aligned}$$

is the equation to the required curve.

(3). To find a spiral, of which the area is always equal to the triangle contained by the tangent, the subtangent, and the radius vector.

The conditions give the equation

$$\int r^2 d\theta = r^2 \frac{d\theta}{dr} = \frac{r^3}{\left(\frac{dr}{d\theta}\right)}.$$

Then, differentiating with respect to θ , we obtain the equation

$$r \frac{d^2 r}{d\theta^2} = 2 \left(\frac{dr}{d\theta}\right)^2,$$

and assuming $r = e^{\int u d\theta}$, this is transformed into $\frac{du}{d\theta} = u^2$,

which gives $u = \frac{1}{c - \theta}$, and $\int u d\theta = \log_c \frac{c_1}{c - \theta}$, therefore the equation to the required curve is

$$r = \frac{c_1}{c - \theta}.$$

(4). To find a curve, such that the solid formed by its revolution about the axis of x shall have its centre of gravity distant from the origin by $\frac{1}{3}$ of the whole axis,

$$\text{we have } \frac{\int y^2 x dx}{\int y^2 dx} = \frac{2}{3} x;$$

therefore, multiplying by $y^2 dx$, and differentiating both sides, we get

$$y^2 x = 2 \int y^2 dx, \therefore 2x \frac{dy}{dx} = y,$$

$$\therefore y^2 = cx.$$

Hence the required curve is a parabola.

(5). An infinity of circles touch one another internally in the same point; it is required to determine a curve which shall cut all these circles at right angles.

Let the point in which the circles touch one another be taken for the origin; then if a be the radius of any one of the circles, the equation to it will be

$$y = \sqrt{2ax - x^2} \dots \dots (1),$$

and if $y = \phi x$ be the equation to the required curve, the tangent of the angle which it makes with the circle will be

$$= \frac{\phi' x - \frac{dy}{dx}}{1 + \phi' x \frac{dy}{dx}},$$

and this is to equal infinity,

$$\therefore 1 + \phi' x \frac{dy}{dx} = 0,$$

$$\text{or } 1 + \phi' x \frac{a - x}{\sqrt{2ax - x^2}} = 0;$$

and eliminating a from this equation, by means of equation (1), we get the differential equation

$$\phi' x = \frac{2xy}{x^2 - y^2} = \frac{dy}{dx},$$

y and x being now the co-ordinates of the required curve.

Putting $\frac{x^2}{2} = u$, and integrating, we have

$$x^2 = 2cy - y^2,$$

which shews that the curve must be a circle with the same origin, and its axis perpendicular to the common axis of the circles touching each other.

(6). To find the bounding curve for a series of ellipses, with coincident axes and equal areas.

Let a and β be the $\frac{1}{2}$ axes of any one of the ellipses; then the last condition gives $a\beta$ constant $= m^2$, suppose; therefore the equation to any one of the ellipses will be

$$y^2 = \frac{m^4}{a^4} (a^2 - x^2),$$

$$\text{or } V = y^2 a^4 - m^4 a^2 + m^4 x^2 = 0.$$

In this equation a is the variable parameter to be eliminated, therefore, differentiating, we have

$$\frac{dV}{da} = 4y^2 a^3 - 2m^4 a, \therefore a^2 = \frac{m^4}{2y^2};$$

and substituting this value in the equation $V = 0$,

$$\text{we have } xy = \frac{m^2}{2}.$$

Hence the required curve is a hyperbola between the asymptotes with its centre in the centre of the ellipses.

THE END.

